

QUANTUM UNIQUE ERGODICITY FOR EISENSTEIN SERIES ON THE HILBERT MODULAR GROUP OVER A TOTALLY REAL FIELD

JIMI L. TRUELSEN

ABSTRACT. W. Luo and P. Sarnak have proved the quantum unique ergodicity property for Eisenstein series on $\mathrm{PSL}(2, \mathbf{Z}) \backslash \mathbf{H}$. Their result is quantitative in the sense that they find the precise asymptotics of the measure considered. We extend their result to Eisenstein series on $\mathrm{PSL}(2, \mathcal{O}) \backslash \mathbf{H}^n$, where \mathcal{O} is the ring of integers in a totally real field of degree n over \mathbf{Q} with narrow class number one, using the Eisenstein series considered by I. Efrat. We also give an expository treatment of the theory of Hecke operators on non-holomorphic Hilbert modular forms.

1. INTRODUCTION

Let \mathbf{H} denote the upper half-plane and Γ be a Fuchsian group of the first kind. We equip the surface $\Gamma \backslash \mathbf{H}$ with the measure induced by the Poincaré metric $d\mu = \frac{dx dy}{y^2}$ on \mathbf{H} . If Γ is hyperbolic we know that the quotient $\Gamma \backslash \mathbf{H}$ is compact and that the Laplace-Beltrami operator Δ associated with this surface, given in local coordinates by $-y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$, has pure point spectrum

$$0 = \lambda_0 < \lambda_1 \leq \dots$$

and that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Inspired by quantum chaos (see [20] and [21] for excellent surveys) Z. Rudnick and P. Sarnak [19] conjectured that

$$(1.1) \quad |\varphi_j|^2 d\mu \rightarrow \frac{1}{\mu(\Gamma \backslash \mathbf{H})} d\mu,$$

where $\{\varphi_j\}$ is an orthonormal basis for $L^2(\Gamma \backslash \mathbf{H})$ of eigenfunctions of Δ with $\Delta \varphi_j = \lambda_j \varphi_j$, and the convergence is in the weak-* topology. This is known as the quantum unique ergodicity conjecture. It has been established by Y. Colin de Verdière [3], A. Shnirelman [22] and S. Zelditch [28] that (1.1) holds for a subsequence of full density.

If $\Gamma = \mathrm{PSL}(2, \mathbf{Z})$ the quotient $\Gamma \backslash \mathbf{H}$ is no longer compact, and Δ does not have pure point spectrum. However, by the Weyl law it is known that

$$\#\{j \in \mathbf{N}_0 \mid |t_j| \leq T\} \sim \frac{\mu(\Gamma \backslash \mathbf{H})}{4\pi} T^2,$$

where $\lambda_j = 1/4 + t_j^2$ are the eigenvalues of Δ . Thus the analogue of the quantum unique ergodicity conjecture is

$$|\varphi_j|^2 d\mu \rightarrow \frac{3}{\pi} d\mu$$

where $\{\varphi_j\}$ is a complete set of orthonormal eigenfunctions of Δ . It was proved in [15] that if the φ_j 's are Hecke eigenforms then the conjecture is true for a (large) subsequence of the full sequence and it has been proved by R. Holowinsky and K. Soundararajan [10] that the conjecture is implied by the Ramanujan-Petterson conjecture.

The author was supported by a stipend (EliteForsk) from The Danish Agency for Science, Technology and Innovation.

In [15] a continuous spectrum analogue of the quantum unique ergodicity conjecture was proved. More precisely it was proved that for $A, B \subset \Gamma \backslash \mathbf{H}$ compact and Jordan measurable, such that $\mu(B) \neq 0$, we have the limit

$$\frac{\int_A |E(z, 1/2 + it)|^2 d\mu}{\int_B |E(z, 1/2 + it)|^2 d\mu} \rightarrow \frac{\mu(A)}{\mu(B)}$$

as $t \rightarrow \infty$, where $E(z, s)$ is the Eisenstein series on $\mathrm{PSL}(2, \mathbf{Z})$. The authors even found explicit asymptotics for the measure $|E(z, 1/2 + it)|^2 d\mu$ (in terms of integration of a continuous function with compact support). In this paper we generalize this result to Eisenstein series $E(z, s, m)$ (it will be defined in Section 11) on $\Gamma \backslash \mathbf{H}^n$, where $\Gamma = \mathrm{PSL}(2, \mathcal{O})$ and \mathcal{O} is the ring of integers in a totally real field K of degree n over \mathbf{Q} with narrow class number one. Note that instead of just one Eisenstein series as in the case of $\mathrm{PSL}(2, \mathbf{Z})$ we have a family of Eisenstein series parametrized by $m \in \mathbf{Z}^{n-1}$.

We investigate the asymptotic behaviour of the measure $d\mu_{m,t} = |E(z, 1/2 + it, m)|^2 d\mu$, where μ is the measure on $\Gamma \backslash \mathbf{H}^n$ induced by the metric $\frac{dx_1 \dots dx_n dy_1 \dots dy_n}{y_1^2 \dots y_n^2}$ on \mathbf{H}^n :

Theorem 1.1. *For $F \in C_c(\Gamma \backslash \mathbf{H}^n)$ we have that*

$$\frac{1}{\log t} \int_{\Gamma \backslash \mathbf{H}^n} F(z) d\mu_{m,t}(z) \rightarrow \frac{\pi^n n R}{2D\zeta_K(2)} \int_{\Gamma \backslash \mathbf{H}^n} F(z) d\mu(z)$$

as $t \rightarrow \infty$, where ζ_K denotes the Dedekind zeta-function and D and R denote the discriminant and regulator of K , respectively.

From this one easily deduces that:

Theorem 1.2. *Let $A, B \subset \Gamma \backslash \mathbf{H}^n$ be compact and Jordan measurable, and assume that $\mu(B) \neq 0$. Then*

$$\frac{\mu_{m,t}(A)}{\mu_{m,t}(B)} \rightarrow \frac{\mu(A)}{\mu(B)}$$

as $t \rightarrow \infty$.

To prove Theorem 1.1 we follow the same strategy as in [15]. The idea in the proof is to find the asymptotics of $\int_{\Gamma \backslash \mathbf{H}^n} f d\mu_{m,t}$, where f is either an incomplete Eisenstein series or a Hecke eigenform, and then use the spectral decomposition of $L^2(\Gamma \backslash \mathbf{H}^n)$. Estimates for various L -functions play a crucial role in the proof, and we will collect these results, as we go along. It should be mentioned that a similar result was shown in the case of a quadratic imaginary field with class number one in [14] using a subconvexity estimate (in the t -aspect) for the standard L -function proved in [18].

I would like to thank my advisor Morten S. Risager for suggesting this problem to me and for excellent guidance and supervision. I would also like to thank Akshay Venkatesh as well as the anonymous referee for useful comments.

2. NOTATION AND TERMINOLOGY

Let K be a totally real field of degree n over \mathbf{Q} and narrow class number one (these are the standard assumptions which are usually made to work with a non-adelic setup in textbooks such as [1] and [6]) and let \mathcal{O} denote the ring of integers in K . Here narrow class number one means that \mathcal{O} is a principal ideal domain and that each non-zero ideal in \mathcal{O} has a generator which is totally positive (this term is explained below).

Let

$$(2.1) \quad \mathrm{Gal}(K/\mathbf{Q}) = \{\psi_1, \dots, \psi_n\}$$

with ψ_1 equal to the identity map on K . In this way we may regard \mathcal{O} as a lattice in \mathbf{R}^n , by the injection $\mathcal{O} \hookrightarrow \mathbf{R}^n$ defined by $a \mapsto (a^{(1)}, \dots, a^{(n)})$, where $a^{(j)} = \psi_j(a)$. Note that this embedding depends on the choice of ordering of the elements in $\text{Gal}(K/\mathbf{Q})$ given in (2.1).

We let \mathcal{O}^\times denote the group of units in \mathcal{O} and $\mathcal{O}^* = \mathcal{O} - \{0\}$. The elements in \mathcal{O}^* for which all the embeddings are positive (such elements are called totally positive) will be denoted \mathcal{O}_+ . We let $\mathcal{O}_+^\times = \mathcal{O}_+ \cap \mathcal{O}^\times$ which clearly is a multiplicative group.

We let \mathcal{D} denote the different, i.e. the inverse ideal of

$$\mathcal{D}^{-1} = \{v \in K \mid \text{Tr}(v\mathcal{O}) \subset \mathbf{Z}\}.$$

It is a well known fact that $\mathcal{D}^{-1} \supset \mathcal{O}$ is a fractional ideal, and since K has narrow class number one there exists $\omega \in \mathcal{O}_+$ such that $\mathcal{D} = (\omega) = \omega\mathcal{O}$ and $\mathcal{D}^{-1} = \omega^{-1}\mathcal{O}$.

It is well known that \mathcal{O} is a free abelian group of rank n , and $\mathcal{O}^\times/\{\pm 1\}$ is a free abelian group of rank $n-1$. In addition we know that for each $u \in \mathcal{O}^\times$ we have $|u^{(1)} \dots u^{(n)}| = 1$. We will assume that $\varepsilon_1, \dots, \varepsilon_{n-1} \in \mathbf{R}_+$ together with -1 generate \mathcal{O}^\times . For later use let

$$(2.2) \quad \begin{pmatrix} e_{1,1} & \cdots & e_{1,n-1} & 1/n \\ \cdots & \cdots & \cdots & \cdots \\ e_{n,1} & \cdots & e_{n,n-1} & 1/n \end{pmatrix} = \begin{pmatrix} \log |\varepsilon_1^{(1)}| & \cdots & \log |\varepsilon_1^{(n)}| \\ \cdots & \cdots & \cdots \\ \log |\varepsilon_{n-1}^{(1)}| & \cdots & \log |\varepsilon_{n-1}^{(n)}| \\ 1 & \cdots & 1 \end{pmatrix}^{-1}.$$

Note that we have the relations

$$(2.3) \quad \sum_{j=1}^n e_{j,q} = 0,$$

and

$$(2.4) \quad \sum_{j=1}^n e_{j,q'} \log |\varepsilon_q^{(j)}| = \delta_{q,q'}$$

for $q, q' = 1, \dots, n-1$.

We let \mathbf{H} denote the upper half plane of \mathbf{C} , i.e.

$$\mathbf{H} = \{z \in \mathbf{C} \mid \text{Im}(z) > 0\}.$$

We will use the convention $z = (z_1, \dots, z_n) \in \mathbf{H}^n$ and $z = (x, y)$ where $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbf{R}_+^n$. Furthermore we will use the notation $dx = dx_1 \dots dx_n$ and $dy = dy_1 \dots dy_n$.

We set $\Gamma = \text{PSL}(2, \mathcal{O}) \subset \text{PSL}(2, \mathbf{R})$. This group is often referred to as the Hilbert modular group. The group Γ does not in general imbed discretely in $\text{PSL}(2, \mathbf{R})$, but it does imbed discretely in $\text{PSL}(2, \mathbf{R})^n$ by the action on \mathbf{H}^n defined by

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \left(\frac{a^{(1)}z_1 + b^{(1)}}{c^{(1)}z_1 + d^{(1)}}, \dots, \frac{a^{(n)}z_n + b^{(n)}}{c^{(n)}z_n + d^{(n)}} \right)$$

which clearly is an extension of the classical action of $\text{PSL}(2, \mathbf{Z})$ on \mathbf{H} by Möbius transformations. For $\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathcal{O})$ we define $\gamma^{(j)} = \pm \begin{pmatrix} a^{(j)} & b^{(j)} \\ c^{(j)} & d^{(j)} \end{pmatrix}$.

If we regard \mathbf{H}^n as a Riemannian manifold with the metric

$$ds^2 = \frac{dx_1^2 + dy_1^2}{y_1^2} + \dots + \frac{dx_n^2 + dy_n^2}{y_n^2}$$

the Laplace-Beltrami operator associated with this metric is

$$\Delta = \Delta_1 + \dots + \Delta_n$$

where $\Delta_j = -y_j^2 \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right)$. In the natural way the metric on \mathbf{H}^n transfers to the quotient $\Gamma \backslash \mathbf{H}^n$. We also see that the Δ_j 's induce symmetric and positive differential operators on $C_b^\infty(\Gamma \backslash \mathbf{H}^n)$ which admit self-adjoint extensions (the Friedrichs extension). It is known that the quotient $\Gamma \backslash \mathbf{H}^n$ has finite volume and as in the case $n = 1$ we will often regard functions on $\Gamma \backslash \mathbf{H}^n$ as functions on the space \mathbf{H}^n which are invariant under Γ . The measure on $\Gamma \backslash \mathbf{H}^n$ induced by the Riemannian metric is denoted μ and one can check that $d\mu = \frac{dx dy}{y_1^2 \dots y_n^2}$ in local coordinates.

3. THE HECKE L -FUNCTION

In the following it will be convenient to set $\rho_j(m) = \pi \sum_{q=1}^{n-1} m_q e_{j,q}$ for $m \in \mathbf{Z}^{n-1}$. Let χ_m denote the following function on \mathbf{C}^{*n} :

$$(3.1) \quad \chi_m(w) = \exp \left(i\pi \sum_{q=1}^{n-1} m_q \sum_{j=1}^n e_{j,q} \log |w_j| \right) = \prod_{j=1}^n |w_j|^{i\rho_j(m)}.$$

Clearly we can regard χ_m as a multiplicative function on \mathcal{O}^* by the usual embedding. For $\beta \in \mathcal{O}_+$ we note that $\chi_m(\beta)$ only depends on the ideal (β) , so in this way we can regard χ_m as a multiplicative function on the non-zero ideals in \mathcal{O} (a so-called Grössencharacter). Note also that for m even, χ_m is trivial on \mathcal{O}^\times . We can now define the Hecke L -function. It is defined by the series

$$\zeta(s, m) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O} \\ \mathfrak{a} \neq 0}} \frac{\chi_m(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s},$$

which converges absolutely for $\text{Re}(s) > 1$, and it can also be written as an Euler product over the prime ideals \mathfrak{p} , i.e.

$$\zeta(s, m) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi_m(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^s} \right)^{-1}.$$

The Hecke L -function has a meromorphic continuation to the entire complex plane. Furthermore $\zeta(s, m)$ is entire if $m \neq 0$. The Dedekind zeta function $\zeta(s, 0)$ (sometimes also denoted ζ_K) has a simple pole at $s = 1$ with residue $\frac{2^{n-1}R}{\sqrt{D}}$ (cf. [1] Section 1.7), and is holomorphic elsewhere. Here $D = \mathcal{N}(\mathcal{D}) = |N(\omega)|$ is the discriminant of K and R is the regulator of K , i.e. the absolute value of the determinant

$$\begin{vmatrix} \log |\varepsilon_1^{(1)}| & \cdots & \log |\varepsilon_1^{(n-1)}| \\ \vdots & \ddots & \vdots \\ \log |\varepsilon_{n-1}^{(1)}| & \cdots & \log |\varepsilon_{n-1}^{(n-1)}| \end{vmatrix}.$$

First we will make a convexity bound for the Hecke L -function on the line $\text{Re}(s) = \sigma$, where $\frac{1}{2} \leq \sigma \leq 1$. It is well known (see [1] Theorem 1.7.2) that the Hecke L -function $\zeta(s, m)$ satisfies the functional equation

$$(3.2) \quad \xi(s, m) = \chi_m(\omega) i^{\text{Tr}(\tau)} \xi(1-s, -m)$$

where $\xi(s, m)$ denotes the completed L -function defined by

$$\xi(s, m) = D^{s/2} \pi^{-ns/2} \zeta(s, m) \prod_{j=1}^n \Gamma \left(\frac{s + \tau_j - i\rho_j(m)}{2} \right),$$

and $\tau = (\tau_1, \dots, \tau_n)$ is a binary vector depending on m with the property that

$$(3.3) \quad \chi_m((\beta)) = \chi_m(\beta) \prod_{j=1}^n \operatorname{sgn}(\beta^{(j)})^{\tau_j}$$

for $\beta \in \mathcal{O}^*$.

Stirling's formula, i.e. the asymptotics of the Γ -function on vertical lines, plays a crucial role in the proof of Theorem 1.1. For any $\sigma \in \mathbf{R}$ we have

$$(3.4) \quad \Gamma(\sigma + it) \sim \sqrt{2\pi} e^{-\pi|t|/2} |t|^{\sigma-1/2}$$

and

$$(3.5) \quad \frac{\Gamma'(\sigma + it)}{\Gamma(\sigma + it)} \sim \log |t|$$

as $|t| \rightarrow \infty$. Using the Phragmén-Lindelöf principle (see [13] Section 5.A), the functional equation (3.2) and Stirling's formula we easily derive the convexity bound

$$(3.6) \quad \zeta(\sigma + it, m) \ll |t|^{\frac{n}{2}(1-\sigma)+\varepsilon}$$

as $|t| \rightarrow \infty$, for any $\varepsilon > 0$ and $\frac{1}{2} \leq \sigma \leq 1$. Note that (3.6) gives the estimate

$$\zeta(1/2 + it, m) \ll |t|^{\frac{n}{4}+\varepsilon}$$

for any $\varepsilon > 0$. For later use it turns out that we need something slightly better ($\frac{n}{4} - \varepsilon$ in the exponent will do), i.e. we need a subconvexity estimate for $\zeta(s, m)$ on the critical line. Such an estimate was proven by P. Söhne [25] (generalizing ideas due to D. R. Heath-Brown [8] and [9]):

Theorem 3.1. *Let $\varepsilon > 0$. Then*

$$\zeta(1/2 + it, m) \ll |t|^{\frac{n}{6}+\varepsilon}$$

as $|t| \rightarrow \infty$.

It is conjectured (and implied by the generalized Riemann hypothesis) that one in fact has

$$\zeta(1/2 + it, m) \ll |t|^\varepsilon$$

for any $\varepsilon > 0$ as $|t| \rightarrow \infty$.

It will also be necessary to estimate the logarithmic derivative of $\zeta(s, m)$ on the line $\operatorname{Re}(s) = 1$. We introduce a von Mangoldt type function on the non-zero ideals in \mathcal{O} defined by

$$\Lambda_m(\mathfrak{a}) = \begin{cases} \chi_m(\mathfrak{a}) \log \mathcal{N}(\mathfrak{p}) & \text{if } \mathfrak{a} = \mathfrak{p}^k \\ 0 & \text{otherwise} \end{cases},$$

where \mathfrak{p} denotes a prime ideal. For $\operatorname{Re}(s) > 1$ we see using the Euler product that

$$\begin{aligned} -\frac{\zeta'(s, m)}{\zeta(s, m)} &= -\sum_{\mathfrak{p}} \left(1 - \frac{\chi_m(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^s}\right) \frac{d}{ds} \left(\frac{1}{1 - \frac{\chi_m(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^s}}\right) \\ &= \sum_{\mathfrak{p}} \frac{\chi_m(\mathfrak{p}) \log \mathcal{N}(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^s \left(1 - \frac{\chi_m(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^s}\right)} \\ &= \sum_{\mathfrak{p}} \log \mathcal{N}(\mathfrak{p}) \sum_{k=1}^{\infty} \frac{\chi_m(\mathfrak{p})^k}{\mathcal{N}(\mathfrak{p})^{sk}} \\ &= \sum_{\mathfrak{a}} \frac{\Lambda_m(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s}. \end{aligned}$$

Thus as in the case of the Riemann zeta-function the logarithmic derivative of $\zeta(s, m)$ can be written as a Dirichlet series.

To estimate the logarithmic derivative of the Hecke L -function we need a zero-free region. By considering exponential sums one can obtain a zero-free region for the Hecke L -function similar to Vinogradov's bound for the Riemann zeta-function (see [27] Chapter 6). This was done by M. Coleman [2]:

Theorem 3.2. *There exist positive constants C and L such that $\zeta(\sigma + it, m) \neq 0$ for $|t| \geq L$ and $\sigma \geq 1 - \frac{C}{(\log |t|)^{2/3} (\log \log |t|)^{1/3}}$.*

At present this is the best zero-free region we know, but the generalized Riemann hypothesis asserts that all zeros of $\zeta(s, m)$ in the critical strip $0 < \operatorname{Re}(s) < 1$ are on the line $\operatorname{Re}(s) = \frac{1}{2}$.

To obtain a sufficiently good estimate for the logarithmic derivative we follow Landau's strategy (cf. [27] Sections 3.9-3.11), which is based on the Borel-Carathéodory theorem. We remark that in order to use this approach it is necessary to estimate the Hecke L -function from below. To this end we consider the following generalization of the Möbius function to non-zero ideals in \mathcal{O} defined by

$$\mu(\mathfrak{p}_1^{\alpha_1} \dots \mathfrak{p}_k^{\alpha_k}) = \begin{cases} (-1)^k & \text{if } \alpha_1, \dots, \alpha_k \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

The function μ has the following property ("Möbius inversion"):

$$(3.7) \quad \sum_{\mathfrak{b} \subset \mathfrak{a}} \mu(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{b} = \mathcal{O} \\ 0 & \text{otherwise} \end{cases},$$

and the proof is the same as in the classical case (see [13] Section 1.3). From this it is clear that

$$(3.8) \quad \frac{1}{\zeta(s, m)} = \sum_{\mathfrak{a}} \frac{\chi_m(\mathfrak{a}) \mu(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s}$$

for $\operatorname{Re}(s) > 1$. Thus

$$\frac{1}{|\zeta(\sigma + it, m)|} \leq \zeta(\sigma, 0)$$

for $\sigma > 1$.

We have the following result due to Landau:

Proposition 3.3. *Let $s = \sigma + it$ and assume that $\zeta(s, m) = O(e^{\varphi(|t|)})$ for $|t| \geq L$ and $1 - \theta(|t|) \leq \sigma \leq 2$ for some positive L , where $\varphi(t)$ and $1/\theta(t)$ are positive increasing functions defined for $t \geq L$ such that $\theta(t) \leq 1$, $\varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$ and $\varphi(t)/\theta(t) = o(e^{\varphi(t)})$. Assume also that there exists a positive constant C such that $\zeta(s, m) \neq 0$ for $|t| \geq L$ and $\sigma \geq 1 - C \frac{\theta(|t|)}{\varphi(|t|)}$. Then*

$$\frac{\zeta'(s, m)}{\zeta(s, m)} = O\left(\frac{\varphi(|t|)}{\theta(|t|)}\right)$$

and

$$\frac{1}{\zeta(s, m)} = O\left(\frac{\varphi(|t|)}{\theta(|t|)}\right)$$

for $|t| \geq L + 1$ and $\sigma \geq 1 - \frac{C\theta(t)}{4\varphi(t)}$.

Using Theorem 3.2 we can apply Proposition 3.3 with $\varphi(t) = (\log t)^{\frac{2}{3}}$ and $\theta(t) = (\log \log t)^{-\frac{1}{3}}$ to obtain the following:

Corollary 3.4. *There exists a positive number L such that for $|t| \geq L$ we have the estimate*

$$\frac{\zeta'(1+it, m)}{\zeta(1+it, m)} = O((\log t)^{\frac{2}{3}}(\log \log t)^{\frac{1}{3}}).$$

In the same way we obtain an explicit lower bound for the Hecke L -function:

Corollary 3.5. *There exists a positive number L such that for $|t| \geq L$ we have the estimate*

$$\frac{1}{\zeta(1+it, m)} = O((\log t)^{\frac{2}{3}}(\log \log t)^{\frac{1}{3}}).$$

4. HECKE OPERATORS

In this section we give an expository treatment of the theory of Hecke operators on non-holomorphic Hilbert modular forms. analogous to the treatment of Hecke operators on holomorphic Hilbert modular forms in [1] Section 1.7 and [6] Section 1.15.

We recall the abstract definition of the Hecke ring (see [23]). We set $G = \mathrm{GL}(2, K)$, $\Gamma = \mathrm{SL}(2, \mathcal{O})$ and let $\mathfrak{D} \subset \mathrm{GL}(2, K)$ denote the 2×2 matrices with entries in \mathcal{O} and totally positive determinant. The Hecke algebra $\mathrm{R}(\Gamma, \mathfrak{D})$ is the \mathbf{C} -vector space of finite formal sums $\sum_k c_k \Gamma \alpha_k \Gamma$, where $\alpha_k \in \mathfrak{D}$ and $c_k \in \mathbf{C}$. The addition in $\mathrm{R}(\Gamma, \mathfrak{D})$ is the obvious one, while the multiplication is defined as follows. Let $\alpha, \beta \in \mathfrak{D}$. It is well known that there exist distinct cosets $\Gamma \alpha_1, \dots, \Gamma \alpha_d$ and $\Gamma \beta_1, \dots, \Gamma \beta_{d'}$, where $\alpha_i, \beta_{i'} \in \mathfrak{D}$, such that $\Gamma \alpha \Gamma = \cup_{i=1}^d \Gamma \alpha_i$ and $\Gamma \beta \Gamma = \cup_{i'=1}^{d'} \Gamma \beta_{i'}$. We define $\Gamma \alpha \Gamma \cdot \Gamma \beta \Gamma = \sum_{i,i'} \Gamma \alpha_i \beta_{i'} \Gamma$, which clearly is independent of the choice of the α_i 's and $\beta_{i'}$'s. We extend this multiplication in the obvious way, making $\mathrm{R}(\Gamma, \mathfrak{D})$ an algebra.

We can define a homomorphism from $\mathrm{GL}(2, \mathbf{R})_+$ to $\mathrm{PSL}(2, \mathbf{R})$ by mapping $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{R})_+$ to $w \mapsto \frac{aw+b}{cw+d}$ in $\mathrm{PSL}(2, \mathbf{R})$. Thus for $w \in \mathbf{H}$ we simply define

$$\tau w = \frac{aw+b}{cw+d}.$$

Therefore we get a natural map from \mathfrak{D} to $\mathrm{PSL}(2, \mathbf{R})^n$ and we see that $\mathrm{R}(\Gamma, \mathfrak{D})$ can be regarded as an algebra of operators on $L^2(\Gamma \backslash \mathbf{H}^n)$ (or even the vector space of automorphic functions) if we define $(\Gamma \alpha \Gamma f)(z) = \sum_{i=1}^d f(\alpha_i z)$.

Two double cosets $\Gamma \alpha \Gamma$ and $\Gamma \beta \Gamma$ are said to be equivalent if $\alpha = \eta \beta$ where $\eta = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$ for some $u \in \mathcal{O}^\times$. Note that if $\alpha = \eta \beta$ then $\alpha^{(j)} z_j = \beta^{(j)} z_j$ for all $j = 1, \dots, n$. Let $\nu \in \mathcal{O}_+$. Inspired by Hecke operators in the case of holomorphic Hilbert modular forms (see [6]) we define

$$(4.1) \quad T_\nu f = \frac{1}{\sqrt{|N(\nu)|}} \sum_{\substack{\det \alpha = u\nu \\ u \in \mathcal{O}_+^\times}} \Gamma \alpha \Gamma f.$$

Here the sum should be taken over inequivalent double cosets.

We can use the class number one assumption to make this more explicit. Consider $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{D}$. Write $a = ra'$ and $c = rc'$ where a' and c' are relative prime (i.e. $(a') + (c') = \mathcal{O}$). There exist $b', d' \in \mathcal{O}$ such that $a'd' - b'c' = 1$ and we see that

$$\begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is upper triangular. Thus for any $\alpha \in \mathfrak{D}$ we can find $\beta \in \mathfrak{D}$, which is upper triangular and satisfies that $\Gamma \alpha = \Gamma \beta$. Using this we can write the Hecke operator as follows

$$(4.2) \quad T_\nu f(z) = \frac{1}{\sqrt{|N(\nu)|}} \sum_{\substack{ad=u\nu \\ u \in \mathcal{O}_+^\times}} \sum_{b \in \mathcal{O}/(d)} f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} z\right).$$

The outer sum is finite by unique factorization and the inner sum is finite since $\mathcal{O}/(d)$ is a finite group. Thus $T_\nu f$ is well defined.

If $\nu, \nu' \in \mathcal{O}_+$ and $\nu = u\nu'$ for some $u \in \mathcal{O}_+^\times$ then by definition $T_\nu = T_{\nu'}$. Thus we define $T_{(\nu)} = T_\nu$. Since we assumed that all ideals have a generator in \mathcal{O}_+ there is a Hecke operator associated with each non-zero ideal. Modifying Theorem 3.12.4 in [7] we obtain that the Hecke operators are self-adjoint.

Now we will investigate the properties of the Hecke operators. We have the following proposition:

Proposition 4.1. *Let $\nu_1, \nu_2 \in \mathcal{O}_+$ be relative prime. Then*

$$T_{\nu_1} T_{\nu_2} = T_{\nu_1 \nu_2}.$$

Proof. Let $f \in L^2(\Gamma \backslash \mathbf{H}^n)$. We have that

$$\begin{aligned} & \sqrt{|N(\nu_1 \nu_2)|} (T_{\nu_1} T_{\nu_2} f)(z) \\ &= \sum_{\substack{a_1 d_1 = u_1 \nu_1 \\ u_1 \in \mathcal{O}_+^\times}} \sum_{\substack{a_2 d_2 = u_2 \nu_2 \\ u_2 \in \mathcal{O}_+^\times}} \sum_{b_1 \in \mathcal{O}/(d_1)} f \left(\begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} z \right) \\ &= \sum_{\substack{a_1 d_1 = u_1 \nu_1 \\ u_1 \in \mathcal{O}_+^\times}} \sum_{\substack{a_2 d_2 = u_2 \nu_2 \\ u_2 \in \mathcal{O}_+^\times}} \sum_{b_1 \in \mathcal{O}/(d_1)} f \left(\begin{pmatrix} a_1 a_2 & b_1 a_2 + d_1 b_2 \\ 0 & d_1 d_2 \end{pmatrix} z \right) \\ &= \sum_{\substack{ad = u \nu_1 \nu_2 \\ u \in \mathcal{O}_+^\times}} \sum_{b \in \mathcal{O}/(d)} f \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} z \right) \\ &= \sqrt{|N(\nu_1 \nu_2)|} (T_{\nu_1 \nu_2} f)(z) \end{aligned}$$

where we have used the Chinese remainder theorem, i.e. that

$$\mathcal{O}/((d_1) \cap (d_2)) \cong \mathcal{O}/(d_1) \oplus \mathcal{O}/(d_2)$$

which holds since $(\nu_1) + (\nu_2) = \mathcal{O}$. □

We need the following important lemma:

Lemma 4.2. *Let $p \in \mathcal{O}_+$ be a prime element. Then for any positive integers k, k' we have*

$$T_{p^k} T_{p^{k'}} = \sum_{d=0}^{\min\{k, k'\}} T_{p^{k+k'-2d}}.$$

Proof. Let $f \in L^2(\Gamma \backslash \mathbf{H}^n)$. We see that

$$\begin{aligned} \sqrt{|N(p^{k+k'})|} (T_{p^k} T_{p^{k'}} f)(z) &= \sum_{\substack{l_1 + l_2 = k \\ l'_1 + l'_2 = k'}} \sum_{\substack{b \in \mathcal{O}/(p^{l_2}) \\ b' \in \mathcal{O}/(p^{l'_2})}} f \left(\begin{pmatrix} p^{l_1} & b \\ 0 & p^{l_2} \end{pmatrix} \begin{pmatrix} p^{l'_1} & b' \\ 0 & p^{l'_2} \end{pmatrix} z \right) \\ &= \sum_{\substack{l_1 + l_2 = k \\ l'_1 + l'_2 = k'}} \sum_{\substack{b \in \mathcal{O}/(p^{l_2}) \\ b' \in \mathcal{O}/(p^{l'_2})}} f \left(\begin{pmatrix} p^{l_1 + l'_1} & bp^{l'_2} + b'p^{l_1} \\ 0 & p^{l_2 + l'_2} \end{pmatrix} z \right). \end{aligned}$$

Removing common factors we get

$$\sum_{d=0}^{\min\{k, k'\}} \sum_{\substack{l_1 + l_2 = k - d \\ l'_1 + l'_2 = k' - d \\ \min\{l'_2, l_1\} = 0}} \sum_{b \in \mathcal{O}/(p^{l_2})} \sum_{b' \in \mathcal{O}/(p^{l'_2 + d})} f \left(\begin{pmatrix} p^{l_1 + l'_1} & bp^{l'_2} + b'p^{l_1} \\ 0 & p^{l_2 + l'_2} \end{pmatrix} z \right).$$

We note that as (b, b') runs over all pairs in $\mathcal{O}/(p^{l_2}) \times \mathcal{O}/(p^{l'_2+d})$ the expression $bp^{l'_2} + b'p^{l_1}$ will assume each value in $\mathcal{O}/(p^{l_2+l'_2})$ exactly $|N(p)|^d$ times. Thus

$$\sqrt{|N(p^{k+k'})|} T_{p^k} T_{p^{k'}} = \sum_{d=0}^{\min\{k, k'\}} |N(p^d)| \sqrt{|N(p^{k+k'-2d})|} T_{p^{k+k'-2d}},$$

and this proves the theorem. \square

Combining Proposition 4.1 and Lemma 4.2 we get:

Theorem 4.3. *Let $(\nu_1), (\nu_2)$ be non-zero ideals in \mathcal{O} . Then*

$$T_{(\nu_1)} T_{(\nu_2)} = \sum_{(d) \supset (\nu_1) + (\nu_2)} T_{(\nu_1)(\nu_2)/(d)^2}.$$

In particular the Hecke operators commute.

From Lemma 4.2 we obtain the following proposition:

Proposition 4.4. *Let $p \in \mathcal{O}_+$ be a prime element. Then for $k \in \mathbf{N}_0$ we have that*

$$(4.3) \quad T_{p^{2k}} = \sum_{l=0}^k (-1)^{k+l} \binom{k+l}{2l} T_p^{2l}$$

and

$$(4.4) \quad T_{p^{2k+1}} = \sum_{l=1}^{k+1} (-1)^{k+l+1} \binom{k+l}{2l-1} T_p^{2l-1}.$$

Proof. We first consider (4.3). The claim certainly holds for $k = 0$ and $k = 1$. Now let $k' \geq 2$ be an integer and assume that the formula holds for $k \leq k'$. Using Lemma 4.2 we get

$$\begin{aligned} T_{p^{2k'+2}} &= T_{p^{2k'}} T_{p^2} - T_{p^{2k'}} - T_{p^{2k'-2}} \\ &= (T_p^2 - 2) T_{p^{2k'}} - T_{p^{2k'-2}} \\ &= T_p^{2k'+2} - (2k' + 1) T_p^{2k'} + (-1)^{k'+1} + \\ &\quad \sum_{l=1}^{k'-1} (-1)^{k'+l+1} \left(2 \binom{k'+l}{2l} + \binom{k'+l-1}{2l-2} - \binom{k'+l-1}{2l} \right) T_p^{2l} \\ &= \sum_{l=0}^{k'+1} (-1)^{k'+l+1} \binom{k'+l+1}{2l} T_p^{2l}. \end{aligned}$$

By induction this proves (4.3), and (4.4) is proved by similar arguments. \square

5. THE FOURIER EXPANSION OF AN AUTOMORPHIC FORM

An automorphic form f is a formal eigenfunction of the Laplacians Δ_j (i.e. f need not be square integrable and we allow f to be identically zero also) which satisfy the growth condition

$$f(z) = o(\exp(2\pi y_j))$$

as $y_j \rightarrow \infty$ for all $j = 1, \dots, n$. This holds in particular if f is square integrable. By construction we have $f(z+l) = f(z)$ for all $l \in \mathcal{O}$. Thus f has a Fourier expansion (see [11]):

Theorem 5.1. *Let f be an automorphic form with Laplace eigenvalues $s_j(1 - s_j)$. Then f admits a Fourier expansion of the form*

$$f(z) = \sum_{l \in \mathcal{O}} a_l(y) e(\text{Tr}(\omega^{-1} l x)),$$

where $e(x) = \exp(2\pi i x)$. Since $f(z)$ is an eigenfunction for the Laplacians $\Delta_1, \dots, \Delta_n$ the l -th Fourier coefficient $a_l(y)$ must satisfy the differential equations

$$(5.1) \quad \frac{\partial^2 a_l(y)}{\partial y_j^2} + \left(\frac{s_j(1 - s_j)}{y_j^2} - 4\pi^2 |(\omega^{-1} l)^{(j)}|^2 \right) a_l(y) = 0$$

for $j = 1, \dots, n$ and hence be of the form

$$a_l(y) = c_l \sqrt{y_1 \dots y_n} \prod_{j=1}^n K_{s_j - \frac{1}{2}}(2\pi |(\omega^{-1} l)^{(j)}| y_j)$$

for $l \neq 0$. The zeroth Fourier coefficient can be written as a linear combination of $\prod_{j=1}^n y_j^{s_j}$ and $\prod_{j=1}^n y_j^{1-s_j}$. Furthermore the coefficients c_l satisfy the bound

$$c_l \ll \exp(\varepsilon |N(l)|)$$

for any $\varepsilon > 0$.

Here K_ν denotes the usual Macdonald Bessel function

$$K_\nu(y) = \frac{1}{2} \int_0^\infty \exp(-y(t + 1/t)/2) t^{\nu-1} dt,$$

which is defined for $y > 0$ and $\nu \in \mathbf{C}$. It is well known that these functions decay exponentially as $y \rightarrow \infty$.

Note that if f is automorphic with respect to Γ then $f(z) = f(uz)$ for $u \in \mathcal{O}_+$, where

$$uz = (u^{(1)} z_1, \dots, u^{(n)} z_n),$$

since all such u 's are squares of units (by the assumption that K has narrow class number one). This implies that $c_l = c_{lu}$ for $l \in \mathcal{O}$ and $u \in \mathcal{O}_+^\times$.

A non-zero square integrable automorphic form f is called a cusp form if

$$(5.2) \quad \int_F f(z) dx = 0.$$

Here

$$F = \{t_1 a_1 + \dots + t_n a_n \mid 0 \leq t_j < 1\}$$

where a_1, \dots, a_n is a \mathbf{Z} -basis for \mathcal{O} and each a_j is regarded as a vector in \mathbf{R}^n by the embedding $a_j \mapsto (a_j^{(1)}, \dots, a_j^{(n)})$. We will refer to F as the fundamental mesh for \mathcal{O} and one can check that the definition of cuspidal is independent of the choice of \mathbf{Z} -basis. By the exponential decay of the Macdonald Bessel function one can deduce that f must be of exponential decay as $y_j \rightarrow \infty$.

Using the Hilbert-Schmidt kernel from [5] Section II.9 one can prove using Lemma I.2.1 in [5] that the vector space of square integrable automorphic forms with given Laplace eigenvalues $\lambda_1, \dots, \lambda_n$ is finite dimensional (see [11] for bounds on the dimensions of the eigenspaces). Now define

$$\iota_j(z) = (z_1, \dots, z_{j-1}, -\overline{z_j}, z_{j+1}, \dots, z_n)$$

for $j = 1, \dots, n$. One easily checks that if f is an automorphic form then so is $f \circ \iota_j$ with the same Laplace eigenvalues. Since the eigenspaces are finite dimensional this means that the eigenvalues of ι_j must be ± 1 . We also see that the Hecke operators, the Δ_j 's and the ι_j 's commute. Furthermore all these operators are self-adjoint. Hence we can choose a

basis for the vector space spanned by cusp forms which consists of cusp forms that are also eigenfunctions for all the Hecke operators and all the ι_j 's. These are called primitive cusp forms. Note that being an eigenfunction of the ι_j 's is simply the same as saying that the function is either even or odd in each x_j .

6. HECKE EIGENVALUES AND AUTOMORPHIC FORMS

In this section we will study automorphic forms which are common eigenfunctions for all the Hecke operators. We first note that the identities derived in Theorem 4.3 and Proposition 4.4 give similar identities for the Hecke eigenvalues:

Theorem 6.1. *Assume that f is a common eigenfunction for all the Hecke operators, i.e. that*

$$T_{(\nu)}f = \lambda((\nu))f$$

for all $\nu \in \mathcal{O}^*$. Then for $\nu_1, \nu_2 \in \mathcal{O}^*$ we have

$$(6.1) \quad \lambda((\nu_1))\lambda((\nu_2)) = \sum_{(d) \supset (\nu_1) + (\nu_2)} \lambda((\nu_1\nu_2/d^2)).$$

For a prime element $p \in \mathcal{O}$ and $k \in \mathbf{N}_0$ we have that

$$(6.2) \quad \lambda((p^{2k})) = \sum_{l=0}^k (-1)^{k+l} \binom{k+l}{2l} \lambda((p))^{2l}$$

and

$$(6.3) \quad \lambda((p^{2k+1})) = \sum_{l=1}^{k+1} (-1)^{k+l+1} \binom{k+l}{2l-1} \lambda((p))^{2l-1}.$$

Using the identities above, we can derive a connection between the Fourier coefficients of $T_{(\nu)}f$ and f , where f is a primitive cusp form:

Theorem 6.2. *Let f be a primitive cusp form with Laplace eigenvalues $s_j(1-s_j)$, and assume that f has the Fourier expansion*

$$f(z) = \sum_{l \in \mathcal{O}^*} c_l \sqrt{y_1 \dots y_n} \left(\prod_{j=1}^n K_{s_j - \frac{1}{2}}(2\pi |(\omega^{-1}l)^{(j)}| y_j) \right) e(\text{Tr}(\omega^{-1}lx)).$$

Then the l -th Fourier coefficient of $T_{(\nu)}f$ is

$$\sum_{\substack{d | \gcd(l', \nu) \\ l'\nu = d^2 l}} c_{l'}$$

for $\nu \in \mathcal{O}_+^*$. In particular $c_{\nu u} = \lambda((\nu))c_u$ for $u \in \mathcal{O}^\times$.

Proof. We apply T_ν on the Fourier expansion

$$\begin{aligned} \sqrt{|N(\nu)|} T_\nu f(z) &= \sum_{l' \in \mathcal{O}^*} c_{l'} \sum_{\substack{ad=uv \\ u \in \mathcal{O}_+^\times}} \sqrt{\frac{|a^{(1)}|}{|d^{(1)}|} y_1 \dots \frac{|a^{(n)}|}{|d^{(n)}|} y_n} \times \\ &\quad \left(\prod_{j=1}^n K_{s_j - \frac{1}{2}}(2\pi |(\omega^{-1}l'a/d)^{(j)}| y_j) \right) \sum_{b \in \mathcal{O}/(d)} e(\text{Tr}(\omega^{-1}l'(ax+b)/d)), \end{aligned}$$

where by abuse of notation

$$(ax+b)/d = ((a^{(1)}x_1 + b^{(1)})/d^{(1)}, \dots, (a^{(n)}x_n + b^{(n)})/d^{(n)}).$$

Now if $d \mid l'$ then

$$\sum_{b \in \mathcal{O}/(d)} e \left(\text{Tr} \left(\omega^{-1} l' \frac{b}{d} \right) \right) = |N(d)|.$$

If $d \nmid l'$ there exist $b' \in \mathcal{O}/(d)$ such that $\text{Tr} \left(\omega^{-1} l' \frac{b'}{d} \right) \notin \mathbf{Z}$. Thus

$$\begin{aligned} \sum_{b \in \mathcal{O}/(d)} e \left(\text{Tr} \left(\omega^{-1} l' \frac{b}{d} \right) \right) &= \sum_{b \in \mathcal{O}/(d)} e \left(\text{Tr} \left(\omega^{-1} l' \frac{b + b'}{d} \right) \right) \\ &= e \left(\text{Tr} \left(\omega^{-1} l' \frac{b'}{d} \right) \right) \times \\ &\quad \sum_{b \in \mathcal{O}/(d)} e \left(\text{Tr} \left(\omega^{-1} l' \frac{b}{d} \right) \right). \end{aligned}$$

But this implies that

$$\sum_{b \in \mathcal{O}/(d)} e \left(\text{Tr} \left(\omega^{-1} l' \frac{b}{d} \right) \right) = 0.$$

Thus

$$\begin{aligned} \sqrt{|N(\nu)|} T_{(\nu)} f(z) &= \sum_{l' \in \mathcal{O}^*} c_{l'} \sum_{\substack{ad=u\nu \\ u \in \mathcal{O}_+^\times}} \sqrt{\frac{|a(1)|}{|d(1)|} y_1 \cdots \frac{|a(n)|}{|d(n)|} y_n} \times \\ &\quad \left(\prod_{j=1}^n K_{s_j - \frac{1}{2}}(2\pi |(\omega^{-1} l' \nu / d^2)^{(j)}| y_j) \right) \times \\ &\quad \sum_{b \in \mathcal{O}/(d)} e \left(\text{Tr} \left(\omega^{-1} l' (ax + b) / d \right) \right) \\ &= \sum_{l' \in \mathcal{O}^*} c_{l'} \sum_{d \mid \gcd(l', \nu)} |N(d)| \sqrt{\frac{|\nu(1)|}{|d(1)|^2} y_1 \cdots \frac{|\nu(n)|}{|d(n)|^2} y_n} \times \\ &\quad \left(\prod_{j=1}^n K_{s_j - \frac{1}{2}}(2\pi |(\omega^{-1} l' \nu / d^2)^{(j)}| y_j) \right) \times \\ &\quad e \left(\text{Tr} \left(\omega^{-1} l' \frac{\nu}{d^2} x \right) \right). \end{aligned}$$

From this it is clear that the l -th Fourier coefficient is

$$\sum_{\substack{d \mid \gcd(l', \nu) \\ l' \nu = d^2 l}} c_{l'}.$$

□

7. THE FUNDAMENTAL DOMAIN FOR Γ_∞

Before we can prove the functional equation for the standard L -function we need a fundamental domain for $\mathcal{O}_+^\times \backslash \mathbf{R}_+^n$ and this immediately gives us a fundamental domain for Γ_∞ as well.

Let F denote the interior of the fundamental mesh of the lattice \mathcal{O} in \mathbf{R}^n given by the embedding defined earlier. Let Γ_∞ denote the stabilizer subgroup at ∞ , i.e.

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} u & l \\ 0 & u^{-1} \end{pmatrix} \mid u \in \mathcal{O}^\times, l \in \mathcal{O} \right\}.$$

From [24] we know the fundamental domain for Γ_∞ :

Proposition 7.1. *The set*

$$F_\infty = \{z \in \mathbf{H}^n \mid x \in F, y \in U_\infty\},$$

is a fundamental domain for Γ_∞ . Here $U_\infty \subset \mathbf{R}_+^n$ is a fundamental domain for $\mathcal{O}_+^\times \backslash \mathbf{R}_+^n$. Explicitly we can choose U_∞ to be the preimage of

$$\mathbf{R}_+ \times [-1, 1]^{n-1} \subset \mathbf{R}_+ \times \mathbf{R}^{n-1}$$

under the map (defined on \mathbf{R}_+^n)

$$y \mapsto \left(\prod_{j=1}^n y_j, \sum_{j=2}^n (e_{j,1} - e_{1,1}) \log \frac{y_j}{\sqrt[n]{\prod_{i=1}^n y_i}}, \dots, \sum_{j=2}^n (e_{j,n-1} - e_{1,n-1}) \log \frac{y_j}{\sqrt[n]{\prod_{i=1}^n y_i}} \right),$$

which is injective.

Let \tilde{y} denote the image of y under the map above. Note that we have the relations

$$(7.1) \quad \sum_{j=2}^n \tilde{y}_j \log |\varepsilon_{j-1}^{(k)}| = \log \frac{y_k}{\sqrt[n]{\tilde{y}_1}}$$

for $k = 2, \dots, n$ which follows since

$$\begin{pmatrix} e_{2,1} & \cdots & e_{2,n-1} \\ \cdots & \cdots & \cdots \\ e_{n,1} & \cdots & e_{n,n-1} \end{pmatrix}^{-1} = \begin{pmatrix} \log |\varepsilon_1^{(2)}| & \cdots & \log |\varepsilon_1^{(n)}| \\ \cdots & \cdots & \cdots \\ \log |\varepsilon_{n-1}^{(2)}| & \cdots & \log |\varepsilon_{n-1}^{(n)}| \end{pmatrix} (I_{n-1} + E_{n-1}).$$

Here I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix and E_{n-1} is the $(n-1) \times (n-1)$ matrix with all entries equal to 1. Inserting (7.1) in (3.1) we get the relation

$$(7.2) \quad \chi_m(y) = \exp \left(i\pi \sum_{q=1}^{n-1} m_q \tilde{y}_{q+1} \right).$$

Note also that by (7.1) the ratios y_j/y_i are bounded.

Later we want to integrate so-called incomplete Eisenstein series. To do so it will be convenient to use the transformation from Proposition 7.1 and for that purpose we need to know the Jacobian determinant:

Lemma 7.2. *The numerical value of the Jacobian determinant of the map in Proposition 7.1 is R^{-1} where R is the regulator of K .*

Proof. Let Ω denote the Jacobian matrix. Note that

$$\frac{\partial \tilde{y}_1}{\partial y_j} = \frac{\tilde{y}_1}{y_j}$$

and

$$\frac{\partial \tilde{y}_{k+1}}{\partial y_j} = \frac{1}{y_j} \sum_{j'=2}^n \delta_{j,j'} (e_{j',k} - e_{1,k}) - \frac{1}{ny_j} \sum_{j'=2}^n (e_{j',k} - e_{1,k})$$

for $k = 1, \dots, n-1$. Thus the y_j 's cancel in the Jacobian determinant and we get

$$\det(\Omega) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ A_1 & e_{2,1} - e_{1,1} + A_1 & \cdots & e_{n,1} - e_{1,1} + A_1 \\ \cdots & \cdots & \cdots & \cdots \\ A_{n-1} & e_{2,n-1} - e_{1,n-1} + A_{n-1} & \cdots & e_{n,n-1} - e_{1,n-1} + A_{n-1} \end{vmatrix}$$

where $A_k = -\frac{1}{n} \sum_{j=2}^n (e_{j,k} - e_{1,k})$ for $1 \leq k \leq n-1$. By recursively subtracting column $j-1$ from column j we do not change the determinant. Expanding by minors in the first row (which has 1 in the first entry and 0 in the other entries) we see that

$$\det(\Omega) = \det([e_{j+1,k} - e_{j,k}]_{1 \leq j,k \leq n-1}).$$

Now using a similar trick on the matrix

$$\begin{pmatrix} e_{1,1} & \cdots & e_{1,n-1} & 1/n \\ \cdots & \cdots & \cdots & \cdots \\ e_{n,1} & \cdots & e_{n,n-1} & 1/n \end{pmatrix} = \begin{pmatrix} \log |\varepsilon_1^{(1)}| & \cdots & \log |\varepsilon_1^{(n)}| \\ \cdots & \cdots & \cdots \\ \log |\varepsilon_{n-1}^{(1)}| & \cdots & \log |\varepsilon_{n-1}^{(n)}| \\ 1 & \cdots & 1 \end{pmatrix}^{-1}$$

recursively subtracting row $k-1$ from row k we see that

$$\pm \frac{\det(\Omega)}{n} = \begin{vmatrix} \log |\varepsilon_1^{(1)}| & \cdots & \log |\varepsilon_1^{(n)}| \\ \cdots & \cdots & \cdots \\ \log |\varepsilon_{n-1}^{(1)}| & \cdots & \log |\varepsilon_{n-1}^{(n)}| \\ 1 & \cdots & 1 \end{vmatrix}^{-1}.$$

But the determinant on the right-hand side is $\pm nR$ (see e.g. [26]). \square

8. THE STANDARD L -FUNCTION

In this section we will consider the L -function associated with a primitive cusp form – the so-called standard L -function – and show that it has a functional equation.

For a primitive cusp form φ with Hecke eigenvalues $\lambda(\mathfrak{a})$ we consider the L -function (defined for $\text{Re}(s) > \frac{3}{2}$)

$$L(s, \varphi, m) = \sum_{\mathfrak{a} \neq 0} \frac{\chi_m(\mathfrak{a}) \lambda(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s}.$$

It should be mentioned that one often uses the notation $L(s, \varphi \otimes \chi_m)$ instead of $L(s, \varphi, m)$.

If we use the relations from Theorem 6.1 we can write $L(s, \varphi, m)$ as the Euler product

$$L(s, \varphi, m) = \prod_{\mathfrak{p}} \frac{1}{1 - \frac{\chi_m(\mathfrak{p}) \lambda(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^s} + \frac{\chi_m(\mathfrak{p})^2}{\mathcal{N}(\mathfrak{p})^{2s}}}$$

where the product is taken over all prime ideals.

Before we go on we need the following result:

Lemma 8.1. *Let f be a formal eigenfunction of the Laplacians $\Delta_1, \dots, \Delta_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Assume that $f(iy) = 0$ for all $y \in \mathbf{R}_+^n$ where $iy = (iy_1, \dots, iy_n)$. Assume also that*

$$(8.1) \quad \frac{\partial f}{\partial x_j}(z_1, \dots, z_{j-1}, iy_j, z_j, \dots, z_n) = 0$$

for all $z_{j'} \in \mathbf{H}$ with $j' \neq j$, $y_j \in \mathbf{R}_+$ and $j = 1, \dots, n$. Then $f(z) = 0$ for all $z \in \mathbf{H}^n$.

Proof. Since f is an eigenfunction of the Δ_j 's which are elliptic differential operators we conclude that f must be real analytic. Hence it suffices to prove that

$$\frac{\partial^{|a+b|} f}{\partial x_1^{a_1} \cdots \partial x_n^{a_n} \partial y_1^{b_1} \cdots \partial y_n^{b_n}}(iy) = 0$$

for all $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbf{N}_0^n$ and $y \in \mathbf{R}_+^n$ – note that we use the notation $|a| = \sum_{j=1}^n a_j$. But clearly this would follow if we could prove that

$$\frac{\partial^{|a|} f}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}(iy) = 0$$

for all $(a_1, \dots, a_n) \in \mathbf{N}_0^n$ and $y \in \mathbf{R}_+^n$.

If $a_j \in \{0, 1\}$ for some j , the result follows immediately from (8.1). Now assume that the result holds if for some j we have $a_j \leq q$, $q \geq 2$. Consider $(a_1, \dots, a_n) \in \mathbf{N}_0^n$ such that $\min\{a_1, \dots, a_n\} = q + 1$; say $a_1 = q + 1$. Then we see that

$$\begin{aligned} \frac{\partial^{|a|} f(iy)}{\partial x_1^{a_1} \dots \partial x_n^{a_n}} &= \frac{\partial^{|a|-2}}{\partial x_1^{a_1-2} \partial x_2^{a_2} \dots \partial x_n^{a_n}} \left(-\frac{1}{y_1^2} \Delta_1 f(iy) - \frac{\partial^2 f(iy)}{\partial y_1^2} \right) \\ &= -\frac{\lambda_1}{y_1^2} \frac{\partial^{|a|-2} f(iy)}{\partial x_1^{a_1-2} \partial x_2^{a_2} \dots \partial x_n^{a_n}} - \frac{\partial^2}{\partial y_1^2} \frac{\partial^{|a|-2} f(iy)}{\partial x_1^{a_1-2} \partial x_2^{a_2} \dots \partial x_n^{a_n}} \\ &= 0 \end{aligned}$$

by induction. This proves the lemma. \square

Now we can extend the holomorphic function $L(s, \varphi, m)$ to an entire function with a functional equation of the usual form:

Theorem 8.2. *Let φ be a primitive cusp form with Laplace eigenvalues $\frac{1}{4} + r_j^2$ and Hecke eigenvalues $\lambda(\mathbf{a})$. Then $L(s, \varphi, m)$ has an analytic continuation to the entire complex plane and it satisfies the functional equation*

$$(8.2) \quad \Lambda(s, \varphi, m) = (-1)^{\text{Tr}(\kappa)} \chi_{2m}(\mathcal{D}) \Lambda(1-s, \varphi, -m)$$

where

$$\begin{aligned} \Lambda(s, \varphi, m) &= D^s \pi^{-ns} L(s, \varphi, m) \times \\ &\quad \prod_{j=1}^n \Gamma\left(\frac{s + \kappa_j + ir_j - i\rho_j(m)}{2}\right) \Gamma\left(\frac{s + \kappa_j - ir_j - i\rho_j(m)}{2}\right) \end{aligned}$$

and $\kappa_j = 0$ if φ is even in x_j and $\kappa_j = 1$ if φ is odd in x_j .

Proof. Consider the function

$$\begin{aligned} f(z) &= \frac{1}{(2\pi i)^{\text{Tr}(\kappa)}} \frac{\partial^{\text{Tr}(\kappa)} \varphi}{\partial x_1^{\kappa_1} \dots \partial x_n^{\kappa_n}}(z) \\ &= \sum_{l \in \mathcal{O}^*} c_l e(\text{Tr}(lx/\omega)) \prod_{j=1}^n \left(\frac{l^{\kappa_j}}{\omega^{\kappa_j}} \right)^{(j)} \sqrt{y_j} K_{ir_j}(2\pi |(l/\omega)^{(j)}| y_j), \end{aligned}$$

which is even in all the x_j -variables. For $\text{Re}(s)$ large (this ensures that we can use the Fourier expansion) consider the integral

$$\begin{aligned} &\frac{\chi_m(\mathcal{D})}{D^s} \int_{\mathcal{O}_+^\times \setminus \mathbf{R}_+^n} f(iy) \prod_{j=1}^n y_j^{s-i\rho_j(m)+\kappa_j-3/2} dy \\ &= \sum_{\mathbf{a} \in \mathcal{O}} \frac{\chi_m(\mathbf{a}) \lambda(\mathbf{a})}{\mathcal{N}(\mathbf{a})^s} \prod_{j=1}^n \int_0^\infty K_{ir_j}(2\pi y_j) y_j^{s-i\rho_j(m)+\kappa_j-1} dy_j \times \\ &\quad \sum_{\beta \in \mathcal{O}_+^\times \setminus \mathcal{O}^\times} c_\beta \prod_{j=1}^n (\text{sgn}(\beta^{(j)}))^{\tau_j} \end{aligned}$$

$$\begin{aligned}
&= L(s, \varphi, m) \prod_{j=1}^n \frac{\Gamma\left(\frac{s+\kappa_j+ir_j-i\rho_j(m)}{2}\right) \Gamma\left(\frac{s+\kappa_j-ir_j-i\rho_j(m)}{2}\right)}{4\pi^{s+\kappa_j-i\rho_j(m)}} \times \\
&\quad \sum_{\beta \in \mathcal{O}_+^\times \backslash \mathcal{O}^\times} c_\beta \prod_{j=1}^n (\text{sgn}(\beta^{(j)}))^{\tau_j}
\end{aligned}$$

where τ is the binary vector satisfying (3.3). Note that we have used the formula (see [1] Lemma 1.9.1)

$$\int_0^\infty K_\nu(2\pi y) y^{s-1} dy = \frac{\Gamma\left(\frac{s+\nu}{2}\right) \Gamma\left(\frac{s-\nu}{2}\right)}{4\pi^s}$$

which is valid for $\text{Re}(s) > |\text{Re}(\nu)|$. That the integral above is convergent follows from the fact that $f(iy) = \frac{(-1)^{\text{Tr}(\kappa)}}{\prod_{j=1}^n y_j^{2\kappa_j}} f(i/y)$ (we use the notation $1/y = (1/y_1, \dots, 1/y_n)$).

If we can prove that

$$(8.3) \quad \sum_{\beta \in \mathcal{O}_+^\times \backslash \mathcal{O}^\times} c_\beta \prod_{j=1}^n (\text{sgn}(\beta^{(j)}))^{\tau_j} \neq 0$$

we have the analytic continuation since

$$\int_{\mathcal{O}_+^\times \backslash \mathbf{R}_+^n} f(iy) \prod_{j=1}^n y_j^{s-i\rho_j(m)+\kappa_j-3/2} dy$$

is an entire function in s (due to exponential decay of f in the y_j -variables). So let us assume that

$$(8.4) \quad \sum_{\beta \in \mathcal{O}_+^\times \backslash \mathcal{O}^\times} c_\beta \prod_{j=1}^n (\text{sgn}(\beta^{(j)}))^{\tau_j} = 0.$$

This implies that the integral considered above vanishes for all s and $m \in \mathbf{Z}^{n-1}$. But using the structure of U_∞ we see that (\tilde{f} is $y \mapsto f(iy)$) composed with the inverse of the map in Proposition 7.1)

$$\begin{aligned}
&\int_{\mathcal{O}_+^\times \backslash \mathbf{R}_+^n} f(iy) \prod_{j=1}^n y_j^{s-i\rho_j(m)+\kappa_j-3/2} dy = \\
&\quad R \int_{-1}^1 \dots \int_{-1}^1 \int_0^\infty \tilde{f}(\tilde{y}) \tilde{y}_1^{s-3/2+\text{Tr}(\kappa)/n} \times \\
&\quad \exp\left(\sum_{q=1}^{n-1} \sum_{j=2}^n (\kappa_{q+1} - \kappa_1) \tilde{y}_j \log |\varepsilon_{j-1}^{(q+1)}|\right) \exp\left(-i\pi \sum_{q=1}^{n-1} m_q \tilde{y}_{q+1}\right) d\tilde{y},
\end{aligned}$$

where we have used (7.2). Since this holds for all m we must have $f(iy) = 0$ for all $y \in \mathbf{R}_+^n$. We also have that f is a formal eigenfunction of the Δ_j 's and since f is even in all the x_j -variables condition (8.1) in Lemma 8.1 is also satisfied. Thus we conclude that f is identically 0. But by the Fourier expansion of f this implies that $c_l = 0$ for all $l \in \mathcal{O}^*$ which contradicts that φ is a primitive cusp form and hence non-zero.

Now we prove the functional equation. As remarked earlier $f(iy) = \frac{(-1)^{\text{Tr}(\kappa)}}{\prod_{j=1}^n y_j^{2\kappa_j}} f(i/y)$. From this one easily deduces that

$$\begin{aligned} \int_{\mathcal{O}_+^\times \backslash \mathbf{R}_+^n} f(iy) \prod_{j=1}^n y_j^{s-i\rho_j(m)+\kappa_j-3/2} dy \\ = (-1)^{\text{Tr}(\kappa)} \int_{\mathcal{O}_+^\times \backslash \mathbf{R}_+^n} f(i/y) \prod_{j=1}^n y_j^{s-i\rho_j(m)-\kappa_j-3/2} dy \\ = (-1)^{\text{Tr}(\kappa)} \int_{\mathcal{O}_+^\times \backslash \mathbf{R}_+^n} f(iy) \prod_{j=1}^n y_j^{i\rho_j(m)-s+\kappa_j-1/2} dy \end{aligned}$$

where we have used that the map $y \mapsto 1/y$ maps a fundamental domain of $\mathcal{O}_+^\times \backslash \mathbf{R}_+^n$ to another fundamental domain. Now (8.2) follows immediately from the calculation above since $\sum_{j=1}^n \rho_j(m) = 0$. \square

Using the Phragmén-Lindelöf principle and the functional equation (8.2) one obtains that

$$L(1/2 + it, \varphi, m) \ll |t|^{\frac{n}{2} + \varepsilon}$$

for any $\varepsilon > 0$ as $|t| \rightarrow \infty$. This is not enough for our purpose, but any improvement in the exponent will do. In the case $K = \mathbf{Q}$ T. Meurman [16] proved that

$$L(1/2 + it, \varphi) \ll \sqrt{r} e^{\pi r/2} |t|^{\frac{1}{3} + \varepsilon},$$

where $\frac{1}{4} + r^2$ is the Laplace eigenvalue and the constant implied only depends on ε . Recently P. Michel and A. Venkatesh [17] and A. Diaconu and P. Garrett [4] proved the estimate that we need in general:

Theorem 8.3. *There exists some $\delta > 0$ such that*

$$L(1/2 + it, \varphi, m) \ll |t|^{\frac{n}{2} - \delta}$$

as $|t| \rightarrow \infty$.

The generalized Riemann hypothesis implies much more, namely that you can take any $\varepsilon > 0$ in the exponent (the Lindelöf hypothesis for the standard L -function). It should be mentioned that the techniques in [18] probably are adequate to provide the subconvexity estimate in Theorem 8.3.

9. THE EISENSTEIN SERIES

In the case where $K = \mathbf{Q}$ we have the Eisenstein series

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \text{Im}(\gamma z)^s.$$

In our case of the Hilbert modular group over general K our candidate for the Eisenstein series would be

$$(9.1) \quad \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \prod_{j=1}^n \text{Im}(\gamma^{(j)} z_j)^{s_j}.$$

Now for this to be well defined we need every term to be independent of the choice of γ in the coset $\Gamma_\infty \backslash \Gamma$. This puts some constraints on the choices of the s_j 's. In fact, for (9.1) to be well defined it is necessary and sufficient that

$$(9.2) \quad |u^{(1)}|^{2s_1} \dots |u^{(n)}|^{2s_n} = 1$$

for all $u \in \mathcal{O}^\times$. The condition (9.2) is certainly equivalent to

$$(9.3) \quad s_1 \log |\varepsilon_j^{(1)}| + \cdots + s_n \log |\varepsilon_j^{(n)}| = i\pi m_j$$

for $j = 1, \dots, n-1$ where $m_j \in \mathbf{Z}$. Let $m = (m_1, \dots, m_{n-1}) \in \mathbf{Z}^{n-1}$ be a fixed vector. If we fix the parameter $s \in \mathbf{C}$ and solve the system of equations

$$\begin{pmatrix} \log |\varepsilon_1^{(1)}| & \cdots & \log |\varepsilon_1^{(n)}| \\ \vdots & \ddots & \vdots \\ \log |\varepsilon_{n-1}^{(1)}| & \cdots & \log |\varepsilon_{n-1}^{(n)}| \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} i\pi m_1 \\ \vdots \\ i\pi m_{n-1} \\ ns \end{pmatrix},$$

we get the solution (cf. (2.2))

$$s_j = s + i\pi \sum_{q=1}^{n-1} m_q e_{j,q} = s + i\rho_j(m)$$

for $j = 1, \dots, n$. From now on we will view s_j as a function of m and s . Thus in conclusion we define the Eisenstein series for Γ as

$$(9.4) \quad E(z, s, m) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \prod_{j=1}^n \text{Im}(\gamma^{(j)} z_j)^{s_j},$$

which is absolutely convergent for $\text{Re}(s) > 1$ (cf. [5] p. 42). It was proved in [5] that $E(z, s, m)$ has a meromorphic continuation to the entire s -plane, and that $E(z, s, m)$ is holomorphic on the line $\text{Re}(s) = 1/2$.

One can verify that the Eisenstein series is an automorphic form with Laplace eigenvalues $s_j(1 - s_j)$ and thus it admits a Fourier expansion. When we calculate the Fourier coefficients it will be convenient to consider the following generalization of the divisor function

$$\sigma_{s,m}(l) = \sum_{\substack{(c) \subset \mathcal{O} \\ c|l}} \chi_{2m}(c) |N(c)|^s.$$

Note that $\sigma_{s,m}$ only depends on the ideal (l) . The Fourier coefficients are known from [5] Section II.2:

Theorem 9.1. *For $l \in \mathcal{O}$ let $a_l(y, s, m)$ denote the l -th Fourier coefficient of $E(z, s, m)$. For $l \neq 0$ we have that*

$$a_l(y, s, m) = \frac{2^n \pi^{ns} \sigma_{1-2s, -m}(l)}{\chi_m(\mathcal{D}) D^s \zeta(2s, -2m)} \prod_{j=1}^n \frac{\sqrt{y_j} K_{s_j - \frac{1}{2}}(2\pi |(l/\omega)^{(j)}| y_j) |l^{(j)}|^{s_j - \frac{1}{2}}}{\Gamma(s_j)}.$$

The zeroth Fourier coefficient is given by

$$a_0(y, s, m) = \left(\prod_{j=1}^n y_j \right)^s \chi_m(y) + \varphi(s, m) \left(\prod_{j=1}^n y_j \right)^{1-s} \chi_{-m}(y)$$

where

$$\varphi(s, m) = \frac{\zeta(2s-1, -2m) \pi^{\frac{n}{2}}}{\zeta(2s, -2m) \sqrt{D}} \prod_{j=1}^n \frac{\Gamma(s_j - \frac{1}{2})}{\Gamma(s_j)}.$$

Note that $\varphi(s, m)$ is unitary for $\text{Re}(s) = \frac{1}{2}$.

As in the classical case we also need to consider incomplete Eisenstein series, i.e. automorphic functions on $\Gamma \backslash \mathbf{H}^n$ formed as Poincaré series which fail to be eigenfunctions of the automorphic Laplacian. Let $h \in C_b^\infty(\mathbf{R}_+)$ and assume that $h(y)y^p \rightarrow 0$ as $y \rightarrow \infty$ and $h(y)y^{-p} \rightarrow 0$ as $y \rightarrow 0$ for all $p \in \mathbf{N}$. For $m \in \mathbf{Z}^{n-1}$ we define

$$(9.5) \quad F(z, h, m) = \sum_{\gamma \in \Gamma^\infty \backslash \Gamma} h \left(\prod_{j=1}^n \text{Im}(\gamma^{(j)} z_j) \right) \prod_{j=1}^n \text{Im}(\gamma^{(j)} z_j)^{i\rho_j(m)}.$$

We will refer to $F(z, h, m)$ as the incomplete Eisenstein series induced by h with parameter m . One easily checks that the incomplete Eisenstein series decay faster than any polynomial in the cusp. In particular they are square integrable since they are bounded. Choosing explicit representatives we see that

$$(9.6) \quad \begin{aligned} F(z, h, 0) &= h \left(\prod_{j=1}^n y_j \right) + h \left(\prod_{j=1}^n \frac{y_j}{x_j^2 + y_j^2} \right) + \\ &\quad \frac{1}{2} \sum_{\substack{c, d \in \mathcal{O}^\times \backslash \mathcal{O}^* \\ \gcd(c, d) = 1}} h \left(\prod_{j=1}^n \frac{y_j}{(c^{(j)} x_j + d^{(j)})^2 + (c^{(j)} y_j)^2} \right). \end{aligned}$$

The following proposition reflects the fact that the Hecke L -function $\zeta(s, m)$ has a pole at $s = 1$ if $m = 0$ but is regular at $s = 1$ if $m \neq 0$:

Proposition 9.2. *For $m \neq 0$ we have*

$$\int_{\Gamma \backslash \mathbf{H}^n} F(z, h, m) d\mu(z) = 0.$$

We also have

$$\int_{\Gamma \backslash \mathbf{H}^n} F(z, h, 0) d\mu(z) = 2^{n-1} R \sqrt{D} \int_0^\infty \frac{h(w)}{w^2} dw.$$

Proof. The last statement follows immediately from change of variables using the injective map from Proposition 7.1 and Lemma 7.2.

The first statement follows from a similar argument. Using again the map from Proposition 7.1 and the relation (7.1) we are lead to consider the integral (which only differs from the integral we wish to compute by scaling with a factor of R)

$$\begin{aligned} &\int_{\mathbf{R}_+ \times [-1, 1]^{n-1}} \frac{h(\tilde{y}_1)}{\tilde{y}_1^2} \exp \left(i\pi \sum_{q=1}^{n-1} m_q \sum_{i=2}^n \tilde{y}_i \sum_{j=1}^n e_{j,q} \log |\varepsilon_{i-1}^{(j)}| \right) d\tilde{y} \\ &= \int_{\mathbf{R}_+ \times [-1, 1]^{n-1}} \frac{h(\tilde{y}_1)}{\tilde{y}_1^2} \exp \left(i\pi \sum_{q=1}^{n-1} m_q \tilde{y}_{q+1} \right) d\tilde{y}. \end{aligned}$$

From this the statement is obvious. \square

The space spanned by incomplete Eisenstein series will be denoted $\mathcal{E}(\Gamma \backslash \mathbf{H}^n)$. Using the transformation from Proposition 9.2 it is clear that the orthogonal complement to $\mathcal{E}(\Gamma \backslash \mathbf{H}^n)$ is the set of functions $f \in L^2(\Gamma \backslash \mathbf{H}^n)$ for which

$$(9.7) \quad \int_F f(z) dx = 0,$$

i.e. the zeroth Fourier coefficient vanishes. As in the classical case $K = \mathbf{Q}$ the space $\mathcal{E}(\Gamma \backslash \mathbf{H}^n)^\perp$ is the closure of the space spanned by cusp forms $\mathcal{C}(\Gamma \backslash \mathbf{H}^n)$ (see [5] Theorem II.9.8). Thus we have the decomposition:

$$(9.8) \quad L^2(\Gamma \backslash \mathbf{H}^n) = \overline{\mathcal{C}(\Gamma \backslash \mathbf{H}^n)} \oplus \overline{\mathcal{E}(\Gamma \backslash \mathbf{H}^n)}.$$

Note that the functions in $\mathcal{C}(\Gamma \backslash \mathbf{H}^n)$ are orthogonal to the constant functions.

10. QUANTUM UNIQUE ERGODICITY

We wish to investigate the behaviour of the measure

$$d\mu_{m,t} = |E(z, 1/2 + it, m)|^2 d\mu$$

as $t \rightarrow \infty$. This is the large eigenvalue limit, since the Laplace eigenvalue of $E(z, 1/2 + it, m)$ is $nt^2 + n/4 + \sum_{j=1}^n \rho_j(m)^2$.

In the subsequent sections we will prove the following two results:

Theorem 10.1. *Consider an incomplete Eisenstein series $F(z, h, k)$. Then we have that*

$$(10.1) \quad \frac{1}{\log t} \int_{\Gamma \backslash \mathbf{H}^n} F(z, h, k) d\mu_{m,t}(z) \rightarrow \frac{\pi^n n R}{2D\zeta(2,0)} \int_{\Gamma \backslash \mathbf{H}^n} F(z, h, k) d\mu(z)$$

as $t \rightarrow \infty$. Note in particular that for $k \neq 0$

$$(10.2) \quad \frac{1}{\log t} \int_{\Gamma \backslash \mathbf{H}^n} F(z, h, k) d\mu_{m,t}(z) \rightarrow 0$$

as $t \rightarrow \infty$, cf. Proposition 9.2.

It is interesting that the asymptotics in (10.1) do not depend on m . The constant $\frac{\pi^n n R}{2D\zeta(2,0)}$ can also be given in terms of the volume, since (see [6])

$$(10.3) \quad \mu(\Gamma \backslash \mathbf{H}^n) = \frac{2\zeta(2,0)D^{\frac{3}{2}}}{\pi^n}.$$

Note that since $\zeta(2) = \frac{\pi^2}{6}$ the result above reduces to the result found by W. Luo and P. Sarnak in [15] for $K = \mathbf{Q}$. The results differ by a factor of 16 – they obtain the asymptotics

$$(10.4) \quad \int_{\Gamma \backslash \mathbf{H}^n} F(z, h) d\mu_t(z) \sim \frac{48}{\pi} \log t \int_{\Gamma \backslash \mathbf{H}^n} F(z, h) d\mu(z)$$

as $t \rightarrow \infty$. This difference is due to a disagreement regarding the value of the integral (12.3) below, which exactly accounts for the factor of 16. In this connection two other errors in [15] should be mentioned. A factor of 2 is missing in the Fourier expansion of the Eisenstein series on page 211. This error is cancelled though since a factor of $\frac{1}{2}$ is missing in front of the logarithmic derivatives of $\Gamma(s/2 \pm it)$ on page 216.

We also obtain the asymptotics for primitive cusp forms:

Theorem 10.2. *Let φ be a primitive cusp form. Then*

$$(10.5) \quad \int_{\Gamma \backslash \mathbf{H}^n} \varphi(z) d\mu_{m,t}(z) \rightarrow 0$$

as $t \rightarrow \infty$.

Combining Theorem 10.1 and Theorem 10.2 we can now prove Theorem 1.1:

Proof of Theorem 1.1. Let $\varepsilon > 0$ be given and set $\Theta = \frac{\pi^n n R}{2D\zeta(2,0)}$. One can prove that the functions which are a sum of a finite number of primitive cusp forms and incomplete Eisenstein series are dense in the space of continuous functions which vanish in the cusp $C_0(\Gamma \backslash \mathbf{H}^n)$ equipped with the sup norm. Hence let $F \in C_c(\Gamma \backslash \mathbf{H}^n)$ and choose primitive

cuspidal forms g_1, \dots, g_k , functions $h_1, \dots, h_l \in C_c^\infty(\mathbf{R}_+)$ and parameters m_1, \dots, m_l such that

$$\|F - G\|_\infty \leq \frac{\varepsilon}{2M\mu(\Gamma \backslash \mathbf{H}^n)},$$

where $G(z) = \sum_{j=1}^k g_j(z) + \sum_{i=1}^l F(z, h_i, m_i)$ and M is a constant depending on the field K – in the case $K = \mathbf{Q}$ one can choose $M = 4$. Now since cuspidal forms decay exponentially in the cusp it follows from (9.6) that we can choose a non-negative $h \in C_c^\infty(\mathbf{R}_+)$ of sufficiently rapid decay such that

$$|F(z) - G(z)| \leq F(z, h, 0) < \frac{\varepsilon}{2\mu(\Gamma \backslash \mathbf{H}^n)}$$

for all $z \in \Gamma \backslash \mathbf{H}^n$. Thus by Theorem 10.1

$$\limsup_{t \rightarrow \infty} \frac{1}{\Theta \log t} \left| \int_{\Gamma \backslash \mathbf{H}^n} (F(z) - G(z)) d\mu_{m,t}(z) \right| < \frac{\varepsilon}{2}.$$

Theorem 10.1 and Theorem 10.2 give us that

$$\lim_{t \rightarrow \infty} \frac{1}{\Theta \log t} \int_{\Gamma \backslash \mathbf{H}^n} G(z) d\mu_{m,t}(z) = \int_{\Gamma \backslash \mathbf{H}^n} G(z) d\mu(z).$$

Hence

$$(10.6) \quad \limsup_{t \rightarrow \infty} \left| \frac{1}{\Theta \log t} \int_{\Gamma \backslash \mathbf{H}^n} F(z) d\mu_{m,t}(z) - \int_{\Gamma \backslash \mathbf{H}^n} F(z) d\mu(z) \right| < \varepsilon.$$

This proves the theorem, since (10.6) holds for any $\varepsilon > 0$. \square

Finally, this enables us to prove the main theorem:

Proof of Theorem 1.2. Let $F, G, f, g \in C_c(\Gamma \backslash \mathbf{H}^n)$ be chosen such that

$$F \geq 1_A \geq f \geq 0$$

and

$$G \geq 1_B \geq g \geq 0,$$

where 1_A denotes the indicator function. Then

$$\frac{\int_{\Gamma \backslash \mathbf{H}^n} f(z) d\mu_{m,t}(z)}{\int_{\Gamma \backslash \mathbf{H}^n} G(z) d\mu_{m,t}(z)} \leq \frac{\mu_{m,t}(A)}{\mu_{m,t}(B)} \leq \frac{\int_{\Gamma \backslash \mathbf{H}^n} F(z) d\mu_{m,t}(z)}{\int_{\Gamma \backslash \mathbf{H}^n} g(z) d\mu_{m,t}(z)}.$$

By Theorem 1.1 we see that

$$\frac{\int_{\Gamma \backslash \mathbf{H}^n} f(z) d\mu(z)}{\int_{\Gamma \backslash \mathbf{H}^n} G(z) d\mu(z)} \leq \liminf_{t \rightarrow \infty} \frac{\mu_{m,t}(A)}{\mu_{m,t}(B)} \leq \limsup_{t \rightarrow \infty} \frac{\mu_{m,t}(A)}{\mu_{m,t}(B)} \leq \frac{\int_{\Gamma \backslash \mathbf{H}^n} F(z) d\mu(z)}{\int_{\Gamma \backslash \mathbf{H}^n} g(z) d\mu(z)}.$$

Since this holds for all F, G, f and g the result follows. \square

11. PROOF OF THEOREM 10.1

Consider $F(z, h, k) \in \mathcal{E}(\Gamma \backslash \mathbf{H}^n)$. By standard unfolding arguments we see that

$$\begin{aligned} \int_{\Gamma \backslash \mathbf{H}^n} F(z, h, k) d\mu_{m,t} \\ &= \int_{\Gamma \backslash \mathbf{H}^n} F(z, h, k) |E(z, 1/2 + it, m)|^2 \frac{dx dy}{y_1^2 \dots y_n^2} \\ &= \int_{U_\infty} h \left(\prod_{j=1}^n y_j \right) \int_F |E(z, 1/2 + it, m)|^2 \frac{dx dy}{\prod_{j=1}^n y_j^{2-i\rho_j(k)}}. \end{aligned}$$

Using the Fourier expansion of the Eisenstein series we get

$$\begin{aligned} \frac{1}{\sqrt{D}} \int_F |E(z, 1/2 + it, m)|^2 dx &= 2 \prod_{j=1}^n y_j + 2 \operatorname{Re} \left(\prod_{j=1}^n y_j^{1+2it} \chi_{2m}(y) \overline{\varphi(1/2 + it, m)} \right) + \\ &\quad \frac{4^n \pi^n \prod_{j=1}^n y_j}{D |\zeta(1 + 2it, -2m)|^2} \sum_{l \in \mathcal{O}^*} |\sigma_{-2it, -m}(l)|^2 \times \\ &\quad \prod_{j=1}^n \frac{|K_{it+i\rho_j(m)}(2\pi |(\omega^{-1}l)^{(j)}| y_j)|^2}{|\Gamma(1/2 + it + i\rho_j(m))|^2}. \end{aligned}$$

Now write

$$\int_{\Gamma \backslash \mathbf{H}^n} F(z, h, k) d\mu_{m,t} = F_1(t) + F_2(t)$$

where

$$\begin{aligned} F_1(t) &= 2\sqrt{D} \int_{U_\infty} h \left(\prod_{j=1}^n y_j \right) \times \\ &\quad \left(\prod_{j=1}^n y_j + \operatorname{Re} \left(\prod_{j=1}^n y_j^{1+2it} \chi_{2m}(y) \overline{\varphi(1/2 + it, m)} \right) \right) \frac{dy}{\prod_{j=1}^n y_j^{2-i\rho_j(k)}} \end{aligned}$$

and

$$\begin{aligned} F_2(t) &= \frac{4^n \pi^n}{\sqrt{D} |\zeta(1 + 2it, -2m)|^2} \sum_{l \in \mathcal{O}^*} \int_{U_\infty} h \left(\prod_{j=1}^n y_j \right) |\sigma_{-2it, -m}(l)|^2 \times \\ &\quad \prod_{j=1}^n \frac{|K_{it+i\rho_j(m)}(2\pi |(\omega^{-1}l)^{(j)}| y_j)|^2}{|\Gamma(1/2 + it + i\rho_j(m))|^2} \frac{dy}{\prod_{j=1}^n y_j^{1-i\rho_j(k)}} \\ &= \frac{4^n \pi^n}{\sqrt{D} |\zeta(1 + 2it, -2m)|^2} \sum_{l \in \mathcal{O}_+^\times \backslash \mathcal{O}^*} \int_{\mathbf{R}_+^n} h \left(\prod_{j=1}^n y_j \right) |\sigma_{-2it, -m}(l)|^2 \times \\ &\quad \prod_{j=1}^n \frac{|K_{it+i\rho_j(m)}(2\pi |(\omega^{-1}l)^{(j)}| y_j)|^2}{|\Gamma(1/2 + it + i\rho_j(m))|^2} \frac{dy}{\prod_{j=1}^n y_j^{1-i\rho_j(k)}}. \end{aligned}$$

It is clear that $F_1(t)$ is a bounded function of t .

Before we go on we need to consider a new L -function. For a purely imaginary we associate to $\sigma_{a,m}$ an L -function which can be computed in terms of $\zeta(s, m)$:

$$\begin{aligned}
\sum_{\mathfrak{a} \neq 0} \frac{\chi_{m'}(\mathfrak{a}) |\sigma_{a,m}(\mathfrak{a})|^2}{\mathcal{N}(\mathfrak{a})^s} &= \prod_{\mathfrak{p}} \sum_{k=0}^{\infty} \frac{\chi_{m'}(\mathfrak{p})^k \sigma_{a,m}(\mathfrak{p}^k) \sigma_{-a,-m}(\mathfrak{p}^k)}{\mathcal{N}(\mathfrak{p})^{ks}} \\
&= \prod_{\mathfrak{p}} \sum_{k=0}^{\infty} \frac{\chi_{m'}(\mathfrak{p})^k}{\mathcal{N}(\mathfrak{p})^{ks}} \frac{1 - \chi_{2m}(\mathfrak{p})^{k+1} \mathcal{N}(\mathfrak{p})^{a(k+1)}}{1 - \chi_{2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^a} \frac{1 - \chi_{-2m}(\mathfrak{p})^{k+1} \mathcal{N}(\mathfrak{p})^{-a(k+1)}}{1 - \chi_{-2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-a}} \\
&= \prod_{\mathfrak{p}} \frac{1}{(1 - \chi_{-2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-a})(1 - \chi_{2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^a)} \times \\
&\quad \sum_{k=0}^{\infty} (2\chi_{m'}(\mathfrak{p})^k \mathcal{N}(\mathfrak{p})^{-sk} - \chi_{m'}(\mathfrak{p})^k \chi_{2m}(\mathfrak{p})^{k+1} \mathcal{N}(\mathfrak{p})^{(a-s)k+a} - \\
&\quad \chi_{m'}(\mathfrak{p})^k \chi_{-2m}(\mathfrak{p})^{k+1} \mathcal{N}(\mathfrak{p})^{-(a+s)k-a}) \\
&= \prod_{\mathfrak{p}} \frac{1}{(1 - \chi_{-2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-a})(1 - \chi_{2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^a)} \times \\
&\quad \left(\frac{2}{1 - \chi_{m'}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-s}} - \frac{\chi_{2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^a}{1 - \chi_{m'+2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{a-s}} - \frac{\chi_{-2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-a}}{1 - \chi_{m'-2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-a-s}} \right) \\
&= \prod_{\mathfrak{p}} \frac{1 + \chi_{m'}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-s}}{(1 - \chi_{m'}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-s})(1 - \chi_{m'+2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{a-s})(1 - \chi_{m'-2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-a-s})} \\
&= \frac{\zeta(s, m')^2 \zeta(s - a, m' + 2m) \zeta(s + a, m' - 2m)}{\zeta(2s, 2m')}.
\end{aligned}$$

To deal with $F_2(t)$ we consider the Mellin transform Mh of h , i.e.

$$(Mh)(r) = \int_0^{\infty} h(w) w^{-r-1} dw.$$

Note that we have the opposite sign convention in the definition of the Mellin transform than the usual one. However, this is also the convention used in [15], and it is the practical one since we then avoid considering $\zeta(-s, m)$ on the left half plane. By the Mellin inversion formula we have

$$h(w) = \frac{1}{2\pi i} \int_{(\sigma)} (Mh)(r) w^r dr$$

for all $\sigma \in \mathbf{R}$. Thus using the L -function we considered earlier we can rewrite the integral $F_2(t)$ as

$$\begin{aligned}
F_2(t) &= \frac{(4\pi)^n}{2\pi i \sqrt{D} |\zeta(1 + 2it, -2m)|^2} \sum_{l \in \mathcal{O}_+^\times \setminus \mathcal{O}^*} \int_{\mathbf{R}_+^n} \int_{(2)} (Mh)(r) |\sigma_{-2it, -m}(l)|^2 \times \\
&\quad \prod_{j=1}^n \frac{|K_{it+i\rho_j(m)}(2\pi|(\omega^{-1}l)^{(j)}|y_j)|^2}{|\Gamma(1/2 + it + i\rho_j(m))|^2} y_j^{i\rho_j(k)+r-1} dr dy
\end{aligned}$$

$$\begin{aligned}
&= \frac{(4\pi)^n}{2\pi i \sqrt{D} |\zeta(1+2it, -2m)|^2 \prod_{j=1}^n |\Gamma(1/2 + it + i\rho_j(m))|^2} \int_{(2)} (Mh)(r) \times \\
&\quad \sum_{l \in \mathcal{O}_+^\times \setminus \mathcal{O}^*} |\sigma_{-2it, -m}(l)|^2 \int_{\mathbf{R}_+^n} \prod_{j=1}^n |K_{it+i\rho_j(m)} \left(2\pi |(l/\omega)^{(j)}| y_j \right)|^2 y_j^{i\rho_j(k)+r-1} dy dr \\
&= \frac{(4\pi)^n}{2\pi i 2^{3n} \sqrt{D} |\zeta(1+2it, -2m)|^2 \prod_{j=1}^n |\Gamma(1/2 + it + i\rho_j(m))|^2} \int_{(2)} (Mh)(r) \times \\
&\quad \sum_{l \in \mathcal{O}_+^\times \setminus \mathcal{O}^*} |\sigma_{-2it, -m}(l)|^2 \prod_{j=1}^n \frac{|\omega^{(j)}|^{i\rho_j(k)+r} \Gamma((i\rho_j(k) + r)/2)^2}{\pi^{i\rho_j(k)+r} |l^{(j)}|^{i\rho_j(k)+r} \Gamma(i\rho_j(k) + r)} \times \\
&\quad \Gamma((i\rho_j(k) + r)/2 + it + i\rho_j(m)) \Gamma((i\rho_j(k) + r)/2 - it - i\rho_j(m)) dr \\
&= \frac{(4\pi)^n}{2\pi i 2^{3n} \sqrt{D} |\zeta(1+2it, -2m)|^2 \prod_{j=1}^n |\Gamma(1/2 + it + i\rho_j(m))|^2} \int_{(2)} B_k(r, t, h) dr
\end{aligned}$$

where

$$\begin{aligned}
B_k(r, t, h) &= (Mh)(r) \frac{\zeta(r, -k)^2 \zeta(r+2it, -k-2m) \zeta(r-2it, -k+2m)}{\zeta(2r, -2k) \pi^{nr}} \times \\
&\quad \prod_{j=1}^n \frac{|\omega^{(j)}|^{i\rho_j(k)+r} \Gamma((i\rho_j(k) + r)/2)^2}{\Gamma(i\rho_j(k) + r)} \times \\
&\quad \Gamma((i\rho_j(k) + r)/2 + it + i\rho_j(m)) \Gamma((r + i\rho_j(k))/2 - it - i\rho_j(m)).
\end{aligned}$$

Note that we have used the fact that for any $b \in \mathbf{R}$ we have the formula (see [12] Section B.4)

$$(11.1) \quad \int_0^\infty |K_{ib}(2\pi t)|^2 t^{s-1} dt = \frac{\Gamma(s/2 + ib) \Gamma(s/2 - ib) \Gamma(s/2)^2}{2^3 \pi^s \Gamma(s)}.$$

Clearly $(Mh)(r)$ is bounded for $\frac{1}{2} \leq \operatorname{Re}(r) \leq 2$ and Γ decays exponentially in vertical strips by Stirling's formula. Furthermore $\zeta(\sigma + it, k)$ is polynomially bounded in t for $\frac{1}{2} \leq \sigma \leq 2$. Hence we can move the integration from the vertical line $\operatorname{Re}(r) = 2$ to the vertical line $\operatorname{Re}(r) = \frac{1}{2}$ perhaps picking up residues from poles at $r = 1$ and $r = 1 \pm 2it$:

$$\begin{aligned}
F_2(t) &= \frac{(\pi/2)^n \int_{(1/2)} B_k(r, t, h) dr}{2\pi i \sqrt{D} |\zeta(1+2it, -2m)|^2 \prod_{j=1}^n |\Gamma(1/2 + it + i\rho_j(m))|^2} + \\
&\quad \frac{(\pi/2)^n \operatorname{res}_{r=1} B_k(r, t, h)}{\sqrt{D} |\zeta(1+2it, -2m)|^2 \prod_{j=1}^n |\Gamma(1/2 + it + i\rho_j(m))|^2} + O(t^{-10})
\end{aligned}$$

where the $O(t^{-10})$ term comes from the possible residues from poles at $r = 1 \pm 2it$, since $(Mh)(\sigma + it)$ is of rapid decay as $t \rightarrow \infty$. Let us evaluate the first term. Since Stirling's formula is no good near the real axis in our case, we have to work around that. Note that for $a, b \in \mathbf{R}$ we have

$$e^{-|a+b|} e^{-|a-b|} \leq e^{-2|a|}.$$

If $|a+b| \geq 1$ and $a \neq 0$ we also have that

$$\frac{1}{|a+b|} \leq \frac{1+|b|}{|a|}.$$

We can now evaluate the first term. Since we are only interested in the asymptotics as $t \rightarrow \infty$ we can assume that $t \geq 1$. Using the subconvexity estimate from Theorem 3.1 and

Stirling's formula we see that ($C_1, C_2, C_3 > 0$ are suitable constants)

$$\begin{aligned} \int_{(1/2)} |B_k(r, t, h)| dr \leq & e^{-\pi t n} t^{-\frac{n}{6} + \varepsilon} C_1 \int_{-\infty}^{\infty} |(Mh)(1/2 + iw)| (1 + |w|)^{\frac{2n}{3} + \varepsilon} dw + \\ & e^{-\pi t n} t^{-\frac{n}{4} + \varepsilon} C_2 \int_{2(t+\rho_j(m)-1)-\rho_j(k)}^{2(t+\rho_j(m)+1)-\rho_j(k)} |(Mh)(1/2 + iw)| dw + \\ & e^{-\pi t n} t^{-\frac{n}{4} + \varepsilon} C_3 \int_{-2(t+\rho_j(m)+1)-\rho_j(k)}^{-2(t+\rho_j(m)-1)-\rho_j(k)} |(Mh)(1/2 + iw)| dw. \end{aligned}$$

Since Mh is of rapid decay the first term dominates, and we obtain the estimate

$$\int_{(1/2)} B_k(r, t, h) dr \ll e^{-t\pi n} |t|^{-\frac{n}{6} + \varepsilon}.$$

By Corollary 3.5 and Stirling's formula we see that

$$\frac{\int_{(1/2)} B_k(r, t, h) dr}{|\zeta(1 + 2it, -2m)|^2 \prod_{j=1}^n |\Gamma(1/2 + it + i\rho_j(m))|^2} \ll |t|^{-\frac{n}{6} + \varepsilon}$$

for any $\varepsilon > 0$.

Now we turn to the residue term. Since $\zeta(s, k)$ is regular at $s = 1$ for $k \neq 0$ the residue term will vanish in this case and we are done. Assume therefore that $k = 0$. We know that

$$\zeta(s, 0) = \frac{\zeta_{-1}}{s-1} + \zeta_0 + O(s-1)$$

and hence

$$\zeta(s, 0)^2 = \frac{\zeta_{-1}^2}{(s-1)^2} + \frac{2\zeta_{-1}\zeta_0}{s-1} + O(1)$$

as $s \rightarrow 1$ where $\zeta_{-1} = \frac{2^{n-1}R}{\sqrt{D}}$ and ζ_0 is some constant. Now introduce $G(r, t, h)$ defined by

$$B_0(r, t, h) = \zeta(r, 0)^2 G(r, t, h).$$

We see that

$$\text{res}_{r=1} B_0(r, t, h) = G(1, t, h) \zeta_{-1} \left(2\zeta_0 + \zeta_{-1} \frac{G'(1, t, h)}{G(1, t, h)} \right).$$

Note that

$$G(1, t, h) = \frac{(Mh)(1) |\zeta(1 - 2it, 2m)|^2}{\zeta(2, 0) \pi^n} D \Gamma(1/2)^{2n} \prod_{j=1}^n |\Gamma(1/2 + it + i\rho_j(m))|^2$$

and

$$\begin{aligned} \frac{G'(1, t, h)}{G(1, t, h)} &= \frac{\zeta'(1 + 2it, -2m)}{\zeta(1 + 2it, -2m)} + \frac{\zeta'(1 - 2it, 2m)}{\zeta(1 - 2it, 2m)} + \\ &\quad \frac{1}{2} \sum_{j=1}^n \left(\frac{\Gamma'(1/2 + it + i\rho_j(m))}{\Gamma(1/2 + it + i\rho_j(m))} + \frac{\Gamma'(1/2 - it - i\rho_j(m))}{\Gamma(1/2 - it - i\rho_j(m))} \right) + C \end{aligned}$$

where C is a constant that does not depend on t . Since

$$(Mh)(1) = \frac{2^{1-n}}{\sqrt{DR}} \int_{\Gamma \backslash \mathbf{H}^n} F(z, h, 0) d\mu(z)$$

by Proposition 9.2 we see using Corollary 3.4 and Stirling's formula that

$$\frac{1}{\log t} F_2(t) \rightarrow \frac{\pi^n n R}{2D\zeta(2,0)} \int_{\Gamma \backslash \mathbf{H}^n} F(z, h, 0) d\mu(z)$$

as $t \rightarrow \infty$.

12. PROOF OF THEOREM 10.2

Let φ be a primitive cusp form with eigenvalues $\frac{1}{4} + r_j^2$ of the Laplacians Δ_j and Hecke eigenvalues $\lambda(\mathfrak{a})$.

We wish to investigate the asymptotic behaviour of the integral

$$(12.1) \quad \int_{\Gamma \backslash \mathbf{H}^n} \varphi(z) d\mu_{m,t} = \int_{\Gamma \backslash \mathbf{H}^n} \varphi(z) E(z, 1/2 + it, m) E(z, 1/2 - it, -m) d\mu$$

where we have used the fact that $\overline{E(z, s, m)} = E(z, \bar{s}, -m)$. To this end we consider the integral

$$(12.2) \quad I(s) = \int_{\Gamma \backslash \mathbf{H}^n} \varphi(z) E(z, 1/2 + it, m) E(z, s, -m) d\mu$$

for $\text{Re}(s) > 1$. We unfold the integral and get using the Fourier expansions of cusp forms and Eisenstein series that

$$\begin{aligned} I(s) &= \int_{F_\infty} \varphi(z) E(z, 1/2 + it, m) \prod_{j=1}^n y_j^{s_j(-m)-2} dx dy \\ &= \frac{2^n \pi^{n(1/2+it)}}{\zeta(1+2it, -2m) \chi_m(\mathcal{D}) D^{it}} \int_{U_\infty} \sum_{l \in \mathcal{O}^*} \sigma_{-2it, -m}(l) c_l \prod_{j=1}^n y_j^{s_j(-m)-1} |l^{(j)}|^{it+i\rho_j(m)} \times \\ &\quad \frac{K_{it+i\rho_j(m)}(2\pi|(l/\omega)^{(j)}|y_j) K_{ir_j}(2\pi|(l/\omega)^{(j)}|y_j)}{\Gamma(1/2 + it + i\rho_j(m))} dy \\ &= \frac{2^n \pi^{n(1/2+it)} (\prod_{j=1}^n |\omega^{(j)}|^{s-i\rho_j(m)})}{\zeta(1+2it, -2m) \chi_m(\mathcal{D}) D^{it} \prod_{j=1}^n \Gamma(1/2 + it + i\rho_j(m))} \sum_{l \in \mathcal{O}_+^\times \backslash \mathcal{O}^*} \chi_{2m}(l) \times \\ &\quad \mathcal{N}((l))^{it-s} \sigma_{-2it, -m}(l) c_l \int_{\mathbf{R}_+^n} \prod_{j=1}^n K_{it+i\rho_j(m)}(2\pi y_j) K_{ir_j}(2\pi y_j) y_j^{s_j(-m)-1} dy. \end{aligned}$$

For $a, b \in \mathbf{R}$ consider the meromorphic function on \mathbf{C} :

$$\Gamma(s, a, b) = \frac{\Gamma((s+ia+ib)/2) \Gamma((s+ia-ib)/2) \Gamma((s-ia-ib)/2) \Gamma((s-ia+ib)/2)}{2^3 \pi^s \Gamma(s)}.$$

It is well known (see [12] Section B.4) that

$$(12.3) \quad \int_0^\infty K_{ia}(2\pi t) K_{ib}(2\pi t) t^{s-1} dt = \Gamma(s, a, b).$$

So we get

$$\begin{aligned} I(s) &= \frac{2^n \pi^{n(1/2+it)}}{\zeta(1+2it, -2m) \chi_m(\mathcal{D}) D^{it}} \prod_{j=1}^n \frac{|\omega^{(j)}|^{s-i\rho_j(m)} \Gamma(s_j(-m), r_j, t + \rho_j(m))}{\Gamma(1/2 + it + i\rho_j(m))} \times \\ &\quad R(s) \sum_{\beta \in \mathcal{O}_+^\times \backslash \mathcal{O}^\times} c_\beta \end{aligned}$$

where

$$\begin{aligned}
R(s) &= \sum_{\mathfrak{a} \in \mathcal{O}} \chi_{2m}(\mathfrak{a}) \mathcal{N}(\mathfrak{a})^{it-s} \sigma_{-2it, -m}(\mathfrak{a}) \lambda(\mathfrak{a}) \\
&= \prod_{\mathfrak{p}} \sum_{k=0}^{\infty} \chi_{2m}(\mathfrak{p})^k \mathcal{N}(\mathfrak{p})^{k(it-s)} \sigma_{-2it, -m}(\mathfrak{p}^k) \lambda(\mathfrak{p}^k) \\
&= \prod_{\mathfrak{p}} \sum_{k=0}^{\infty} \chi_{2m}(\mathfrak{p})^k \mathcal{N}(\mathfrak{p})^{k(it-s)} \lambda(\mathfrak{p}^k) \sum_{j=0}^k \chi_{-2m}(\mathfrak{p})^j \mathcal{N}(\mathfrak{p})^{-2ijt} \\
&= \prod_{\mathfrak{p}} \sum_{k=0}^{\infty} \chi_{2m}(\mathfrak{p})^k \mathcal{N}(\mathfrak{p})^{k(it-s)} \lambda(\mathfrak{p}^k) \frac{1 - \chi_{-2m}(\mathfrak{p})^{k+1} \mathcal{N}(\mathfrak{p})^{-2(k+1)it}}{1 - \chi_{-2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-2it}} \\
&= \prod_{\mathfrak{p}} \frac{1}{1 - \chi_{-2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-2it}} \left(\sum_{k=0}^{\infty} \chi_{2m}(\mathfrak{p})^k \mathcal{N}(\mathfrak{p})^{k(it-s)} \lambda(\mathfrak{p}^k) - \right. \\
&\quad \left. \chi_{-2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-2it} \sum_{k=0}^{\infty} \lambda(\mathfrak{p}^k) \mathcal{N}(\mathfrak{p})^{k(-it-s)} \right) \\
&= \prod_{\mathfrak{p}} \frac{1}{1 - \chi_{-2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-2it}} \times \\
&\quad \left(\frac{1}{1 - \lambda(\mathfrak{p}) \chi_{2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{it-s} + \chi_{2m}(\mathfrak{p})^2 \mathcal{N}(\mathfrak{p})^{2(it-s)}} - \right. \\
&\quad \left. \frac{\chi_{-2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-2it}}{1 - \lambda(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-it-s} + \mathcal{N}(\mathfrak{p})^{2(-it-s)}} \right) \\
&= \prod_{\mathfrak{p}} \frac{1 - \chi_{2m}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-2s}}{(1 - \chi_{2m}(\mathfrak{p}) \lambda(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{it-s} + \chi_{2m}(\mathfrak{p})^2 \mathcal{N}(\mathfrak{p})^{2(it-s)})} \times \\
&\quad \frac{1}{(1 - \lambda(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-it-s} + \mathcal{N}(\mathfrak{p})^{2(-it-s)})} \\
&= \frac{L(s - it, \varphi, 2m) L(s + it, \varphi, 0)}{\zeta(2s, 2m)}.
\end{aligned}$$

From this we see that $I(s)$ has an analytic continuation to the entire s -plane, and we wish to investigate the asymptotic behaviour of $I(1/2 - it)$ as $t \rightarrow \infty$. From Stirling's formula we deduce that

$$\prod_{j=1}^n \frac{\Gamma(1/2 - it - i\rho_j(m), r_j, t + \rho_j(m))}{\Gamma(1/2 + it + i\rho_j(m))} \ll |t|^{-n/2}$$

as $t \rightarrow \infty$. Using Proposition 3.5 the proof of Theorem 10.2 boils down to proving a subconvexity estimate for $L(s, \varphi, 2m)$ on the line $\operatorname{Re}(s) = \frac{1}{2}$. More precisely we need the estimate

$$L(1/2 + it, \varphi, 2m) \ll |t|^{\frac{n}{2} - \delta}$$

as $|t| \rightarrow \infty$ for some $\delta > 0$, and this follows from Theorem 8.3. Note that if φ is odd then $I(1/2 - it) = 0$, since $L(1/2, \varphi, 0) = 0$ by the functional equation.

REFERENCES

- [1] D. Bump. *Automorphic forms and representations*. Cambridge University Press (1997).
- [2] M. D. Coleman. *A zero-free region for the Hecke L -functions*. *Mathematika* **37** (1990), no. 2, 287-304.

- [3] Y. Colin de Verdière. *Ergodicité et fonctions propre du laplacien*. Comm. Math. Phys. **102** (1985), no. 3, 497-502.
- [4] A. Diaconu and P. Garrett. *Subconvexity bounds for automorphic L -functions*. Preprint (2008).
- [5] I. Y. Efrat. *The Selberg trace formula for $\mathrm{PSL}_2(\mathbf{R})^n$* . Mem. Amer. Math. Soc. **65** (1987).
- [6] P. Garrett. *Holomorphic Hilbert modular forms*. Wadsworth & Brooks (1990).
- [7] D. Goldfeld. *Automorphic forms and L -functions for the group $\mathrm{GL}(n, \mathbf{R})$* . Cambridge University Press (2006).
- [8] D. R. Heath-Brown. *Hybrid bounds for Dirichlet L -functions*. Invent. Math. **47** (1978), no. 2, 149-170.
- [9] D. R. Heath-Brown. *The growth rate of the Dedekind zeta-function on the critical line*. Acta Arith. **49** (1988), no. 4, 323-339.
- [10] R. Holowinsky and K. Soundararajan. *Mass equidistribution for Hecke eigenforms*. Preprint (2008).
- [11] J. Huntley. *Spectral multiplicity on products of hyperbolic spaces*. Proc. Amer. Math. Soc. **111** (1991), no. 1, 1-12.
- [12] H. Iwaniec. *Spectral methods of automorphic forms*. AMS, 2nd edition (2002).
- [13] H. Iwaniec and E. Kowalski. *Analytic number theory*. AMS (2004).
- [14] S. Koyama. *Quantum ergodicity of Eisenstein series for arithmetic 3-manifolds*. Comm. Math. Phys. **215** (2000), no. 2, 477-486.
- [15] W. Luo and P. Sarnak. *Quantum ergodicity of eigenfunctions on $\mathrm{PSL}_2(\mathbf{Z}) \backslash \mathbf{H}^2$* . Pub. Math. l'I.H.É.S. **81** (1995), 207-237.
- [16] T. Meurman. *On the order of the Maass L -function on the critical line*. Colloq. Math. Soc. Janos Bolyai **51** (1990), 325-354.
- [17] P. Michel and A. Venkatesh. In preparation.
- [18] Y. Petridis and P. Sarnak. *Quantum unique ergodicity for $\mathrm{SL}_2(\mathcal{O}) \backslash \mathbf{H}^3$ and estimates for L -functions*. J. Evol. Equa. **1** (2001), no. 3, 277-290.
- [19] Z. Rudnick and P. Sarnak. *The behaviour of eigenstates of arithmetic hyperbolic manifolds*. Comm. Math. Phys. **161** (1994), no. 1, 195-213.
- [20] P. Sarnak. *Arithmetic quantum chaos*. Israel Math. Conf. Proc. **8** (1995), 183-236.
- [21] P. Sarnak. *Spectra of hyperbolic surfaces*. Bull. Amer. Math. Soc. **40** (2003), no. 4, 441-478.
- [22] A. Shnirelman. *Ergodic properties of eigenfunctions*. Usp. Mat. Nauk. **29** (1974), no. 6, 181-182.
- [23] G. Shimura. *Introduction to the arithmetic theory of automorphic functions*. Princeton University Press (1994).
- [24] C. L. Siegel. *Lectures on advanced analytic number theory*. Tata Institute of Fundamental Research, Bombay (1965).
- [25] P. Söhne. *An upper bound for the Hecke zeta-functions with Groessencharacters*. J. Number Theory **66** (1997), no. 2, 225-250.
- [26] C. M. Sorensen. *Fourier expansions of the Eisenstein series on the Hilbert modular group and Hilbert class fields*. Trans. Amer. Math. Soc. **354** (2002), no. 12, 4847-4869.
- [27] E. C. Titchmarsh. *The theory of the Riemann zeta-function*. Oxford University Press (1986).
- [28] S. Zelditch. *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*. Duke Math. J. **55** (1987), no. 4, 919-941.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF AARHUS, NY MUNKEGADE BUILDING
 1530, 8000 AARHUS C, DENMARK
E-mail address: lee@imf.au.dk